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TESIS DOCTORAL

On Algebraic Properties of some q-Multiple Orthogonal Polynomials

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To my family

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Chapter 1

Introduction

1.1 Thesis structure

After an introductory discussion, in which the main notions and background materials of discrete multiorthogonality are addressed, we focus our attention on some new results, namely Chapters 2, 3, and 4. These three chapters constitute the main core of the Thesis. Indeed, Chapter 2 is focused on the study of four new families of q-multiple orthogonal polynomials, namely q-multiple Charlier, q-multiple Meixner of the first and second kind, respectively, and q-multiple Kravchuk. The raising operators and Rodrigues-type formulas, which provide an explicit expression for these new families, are obtained. Chapter 3 contains a detailed study of some algebraic properties for the aforementioned q-families of multiple orthogonal polynomials. More specifically, the $(r+1)$ order recurrence relation as well as the $(r+1)$ order difference equations in the discrete variable on the real line are obtained. Here the letter r is used to denote the dimension of the vector measure $\vec{\mu}$. Finally, in Chapter 4 some limit relations between the attained q-families of multiple orthogonal polynomials (when the parameter q approaches 1) and discrete multiple orthogonal polynomials are established.

1.2 Discrete multiple orthogonal polynomials

In this Thesis we will avoid any discussion on scalar orthogonal polynomials; that is, when $r = 1$. This situation is naturally included in our study. Furthermore, there is a vast literature on this topic both from analytic and applied point of view. Some well-known materials that we have used as basic references on classical properties of scalar orthogonal polynomials are: [1, 38, 43, 46, 61, 62], these references constitute the main sources for known algebraic relations and properties of orthogonal polynomials while [26, 45, 51, 74, 77, 90] serve as a general theory for orthogonal polynomials.

During the last couple of decades the notion of multiple orthogonal polynomials has received special attention both in pure and applied mathematics (see [2, 6, 19, 85] for theoretical aspects and [20, 21, 28, 42, 54, 64] for a more applied perspective). Multiple orthogonal polynomials have been a subject of investigation for a while [4, 24, 29, 30, 55, 68], they are polynomials that satisfy orthogonal conditions shared with respect to a set of measures (see [25, 47, 86]). Such polynomials were first introduced by Hermite in his proof of the transcendence of the number e, and were subsequently used in number theory and approximation theory (see [5, 7, 56, 76, 82, 87]). Indeed, they are related to the simultaneous rational approximation of a system of r analytic functions [57, 88]. However, nowadays, only few concrete examples of q-multiple orthogonal polynomials have been obtained [11, 13, 81] in contrast with other multiple orthogonal polynomial families (see [7, 10, 16, 17, 32, 33, 94, 95]).

Let $\vec{\mu} = (\mu_1, \ldots, \mu_r)$ be a vector of r positive Borel measures supported on $\mathbb{R}$, and let $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$ be a multi-index. By $\mathbb{N}$ we denote the set of all nonnegative integers. A type II multiple orthogonal polynomial $P_{\vec{n}}$, corresponding to the multi-index $\vec{n}$, is a polynomial of degree $\leq |\vec{n}| = n_1 + \cdots + n_r$ which satisfies the orthogonality conditions [77]

$$\int_{\Omega_i} P_{\vec{n}}(x) x^k d\mu_i(x) = 0, \qquad k = 0, \ldots, n_i - 1, \qquad i = 1, \ldots, r, \tag{1.2.1}$$

where Ω_i is the smallest interval that contains $\operatorname{supp}(\mu_i)$. In this thesis we will consider the situation when $P_{\vec{n}}$ is a monic multiple orthogonal polynomial and has exactly degree $|\vec{n}|$. If the measures in (1.2.1) are discrete

$$\mu_i = \sum_{k=0}^{N_i} \omega_{i,k} \delta_{x_{i,k}}, \qquad \omega_{i,k} > 0, \qquad x_{i,k} \in \mathbb{R}, \qquad N_i \in \mathbb{N} \cup \{+\infty\}, \qquad i = 1, 2, \ldots, r, \tag{1.2.2}$$

where $\delta_{x_{i,k}}$ denotes the Dirac delta function and $x_{i_1,k} \neq x_{i_2,k}$, $k = 0, \ldots, N_i$, whenever $i_1 \neq i_2$, the corresponding polynomial solution of the above linear system of equations (1.2.1) is called discrete multiple orthogonal polynomial.

For the discrete measures (1.2.2) we have that $\operatorname{supp}(\mu_i)$ is the closure of $\{x_{i,k}\}_{k=0}^{N_i}$ and that Ω_i is the smallest closed interval on $\mathbb{R}$ which contains $\{x_{i,k}\}_{k=0}^{N_i}$. Moreover, the above orthogonality conditions (1.2.1) give a linear system of $|\vec{n}|$ homogeneous equations for the $|\vec{n}| + 1$ unknown coefficients of $P_{\vec{n}}(x)$ an the aforementioned polynomial solution $P_{\vec{n}}$ always exists. We focus our attention on a unique solution (up to a multiplicative factor) with $\deg P_{\vec{n}}(x) = |\vec{n}|$. If this happens for every multi-index $\vec{n}$, we say that $\vec{n}$ is normal [77]. If the above system of measures forms an AT system [77], then every multi-index is normal. Indeed, we will deal with such a system of discrete measures, where $\Omega_i = \Omega \in \mathbb{R}^+$ for each $i = 1, 2, \ldots, r$.

Definition 1.2.1. The system of positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$, given in (1.2.2), forms an AT system if there exist r continuous functions $\upsilon_1, \ldots, \upsilon_r$ on Ω with $\upsilon_i(x_k) = \omega_{i,k}$, $k = 0, \ldots, N_i$, $i = 1, 2, \ldots, r$, such that the $|\vec{n}|$ functions

$$\upsilon_1(x), x\upsilon_1(x), \ldots, x^{n_1-1}\upsilon_1(x), \ldots, \upsilon_r(x), x\upsilon_r(x), \ldots, x^{n_r-1}\upsilon_r(x),$$

form a Chebyshev system on Ω for each multi-index $\vec{n}$ with $|\vec{n}| < N + 1$, i.e., every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x)\upsilon_i(x)$, where $Q_{n_i-1} \in \mathbb{P}_{n_i-1} \setminus \{0\}$, has at most $|\vec{n}| - 1$ zeros on Ω.

1.2.1 Multiple Charlier polynomials

The monic multiple Charlier polynomials [10] $C_{\vec{n}}^{\vec{\alpha}}(x)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ satisfy the following orthogonality conditions with respect to r Poisson distributions with different positive parameters $\alpha_1, \ldots, \alpha_r$ (indexed by $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r)$)

$$\sum_{x=0}^{\infty} C_{\vec{n}}^{\vec{\alpha}}(x)(-x)_j \upsilon^{\alpha_i}(x) = 0, \qquad j = 0, \ldots, n_i - 1, \qquad i = 1, \ldots, r,$$

where

$$\upsilon^{\alpha_i}(x) = \begin{cases} \dfrac{\alpha_i^x}{\Gamma(x+1)}, & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Z}^-, \\ 0, & \text{otherwise,} \end{cases}$$

and $(x)_j = (x)(x+1)\cdots(x+j-1)$, $(x)_0 = 1$, $j \geq 1$, denotes the Pochhammer symbol.

In [10] the authors consider a normal multi-index $\vec{n} \in \mathbb{N}^r$, whenever $\alpha_i > 0$, $i = 1, 2, \ldots, r$, and with all the α_i different. Moreover, it was found the following raising operators

$$\mathcal{L}^{\alpha_i}\left[C_{\vec{n}}^{\vec{\alpha}}(x)\right] = -C_{\vec{n}+\vec{e}_i}^{\vec{\alpha}}(x), \qquad i = 1, \ldots, r, \tag{1.2.3}$$

where

$$\mathcal{L}^{\alpha_i} \stackrel{\text{def}}{=} \frac{\alpha_i}{\upsilon^{\alpha_i}(x)} \bigtriangledown \upsilon^{\alpha_i}(x),$$

and $\bigtriangledown f(x) = f(x) - f(x-1)$ denotes the backward difference operator. As a consequence of (1.2.3) the Rodrigues-type formula

$$C_{\vec{n}}^{\vec{\alpha}}(x) = (-1)^{|\vec{n}|} \left(\prod_{i=1}^{r} \alpha_i^{n_i}\right) \Gamma(x+1) \mathcal{C}_{\vec{n}}^{\vec{\alpha}} \left(\frac{1}{\Gamma(x+1)}\right), \tag{1.2.4}$$

holds, where

$$\mathcal{C}_{\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r} \left(\alpha_i^{-x} \bigtriangledown^{n_i} \alpha_i^{x}\right).$$

Two important algebraic structural properties are known for multiple Charlier polynomials [10], namely the $(r+1)$-order linear difference equation [65]

$$\prod_{i=1}^{r} \mathcal{L}^{\alpha_i}\left[\bigtriangleup C_{\vec{n}}^{\vec{\alpha}}(x)\right] = -\sum_{i=1}^{r} n_i \prod_{\substack{j=1 \\ j \neq i}}^{r} \mathcal{L}^{\alpha_j}\left[C_{\vec{n}}^{\vec{\alpha}}(x)\right], \tag{1.2.5}$$

where

$$\bigtriangleup f(x) = f(x+1) - f(x), \quad \text{is the forward difference operator,}$$

and the recurrence relation [10, 50]

$$xC_{\vec{n}}^{\vec{\alpha}}(x) = C_{\vec{n}+\vec{e_k}}^{\vec{\alpha}}(x) + (|\vec{n}| + \alpha_k)C_{\vec{n}}^{\vec{\alpha}}(x) + \sum_{i=1}^{r} \alpha_i n_i C_{\vec{n}-\vec{e}_i}^{\vec{\alpha}}(x), \tag{1.2.6}$$

where the multi-index $\vec{e}_i$ is the standard r dimensional unit vector with the i-th entry equals 1 and 0 otherwise.

Interestingly, the multiple Charlier polynomials $C_{\vec{n}}^{\vec{\alpha}}(x)$ are common eigenfunctions of the above two linear difference operators of order $(r+1)$, namely (1.2.5) and (1.2.6).

Also, in [10] was given the second of Appell's hypergeometric functions of two variables [38].

$$\begin{aligned} C_{n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} \sum_{k=0}^{n_1}\sum_{l=0}^{n_2}(-n_1)_k(-n_2)_l(-x)_{k+l}\frac{(1/\alpha_1)^k}{k!}\frac{(1/\alpha_2)^l}{l!} \\ &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} \lim_{\gamma\to+\infty} F_2\left(-x,-n_1,-n_2;\gamma,\gamma;-\frac{\gamma}{\alpha_1},-\frac{\gamma}{\alpha_2}\right), \end{aligned}$$

where

$$F_2(\alpha,\beta,\beta';\gamma,\gamma';x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}x^m y^n.$$

1.2.2 Multiple Meixner polynomials of the first kind

The monic multiple Meixner polynomials of the first kind [10] $M_{\vec{n}}^{\vec{\alpha},\beta}(x)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ satisfy the following orthogonality conditions with different positive parameters $\alpha_1,\ldots,\alpha_r$ (indexed by $\vec{\alpha} = (\alpha_1,\ldots,\alpha_r)$) and $\beta > 0$

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{\vec{\alpha},\beta}(x)(-x)_j v^{\alpha_i,\beta}(x) = 0, \qquad j = 0,\ldots,n_i-1, \qquad i = 1,\ldots,r,$$

where

$$v^{\alpha_i,\beta}(x) = \begin{cases} \dfrac{\Gamma(\beta+x)}{\Gamma(\beta)}\dfrac{\alpha_i^x}{\Gamma(x+1)}, & \text{if} \quad x \in \mathbb{R}\setminus(\mathbb{Z}^- \cup \{-\beta,-\beta-1,\beta-2,\ldots\}), \\ 0, & \text{otherwise.} \end{cases}$$

In [10] the authors consider a normal multi-index $\vec{n} \in \mathbb{N}^r$, whenever $0 < \alpha_i < 1$, $i = 1,2,\ldots,r$, and with all the α_i different. Moreover, it was found the following raising operators

$$\mathcal{L}^{\alpha_i,\beta}\left[M_{\vec{n}}^{\vec{\alpha},\beta}(x)\right] = -M_{\vec{n}+\vec{e}_i}^{\vec{\alpha},\beta-1}(x), \qquad i = 1,\ldots,r, \tag{1.2.7}$$

where

$$\mathcal{L}^{\alpha_i,\beta} \stackrel{\text{def}}{=} \frac{\alpha_i(\beta-1)}{(1-\alpha_i)\,v^{\alpha_i,\beta-1}(x)} \nabla\, v^{\alpha_i,\beta}(x).$$

As a consequence of (1.2.7) there holds the Rodrigues-type formula

$$M_{\vec{n}}^{\vec{\alpha},\beta}(x) = (\beta)_{|\vec{n}|}\left[\prod_{i=1}^{r}\left(\frac{\alpha_i}{\alpha_i-1}\right)^{n_i}\right]\frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta+x)}\mathcal{M}_{\vec{n}}^{\vec{\alpha}}\left(\frac{\Gamma(\beta+|\vec{n}|+x)}{\Gamma(\beta+|\vec{n}|)\Gamma(x+1)}\right), \tag{1.2.8}$$

where

$$\mathcal{M}_{\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r}\left(\alpha_i^{-x}\,\nabla^{n_i}\,\alpha_i^{x}\right).$$

Two important algebraic structural properties are known for multiple Meixner polynomials of the first kind [10], namely the $(r+1)$-order linear difference equation [65]

$$\prod_{i=1}^{r}\mathcal{L}^{\alpha_i,\beta+i+1-r}\left[\,\triangle\, M_{\vec{n}}^{\vec{\alpha},\beta}(x)\right] = -\sum_{i=1}^{r} n_i\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{L}^{\alpha_j,\beta+j+1-r}\left[M_{\vec{n}}^{\vec{\alpha},\beta}(x)\right], \tag{1.2.9}$$

and the recurrence relation [10, 50]

$$\begin{aligned} xM_{\vec{n}}^{\vec{\alpha},\beta}(x) = M_{\vec{n}+\vec{e_k}}^{\vec{\alpha},\beta}(x) + \left[(\beta+|\vec{n}|)\left(\frac{\alpha_k}{1-\alpha_k}\right)+\sum_{i=1}^{r}\frac{n_i}{1-\alpha_i}\right]M_{\vec{n}}^{\vec{\alpha}\beta}(x) \\ +\sum_{i=1}^{r}\frac{\alpha_i n_i\left(\beta+|\vec{n}|-1\right)}{\left(\alpha_i-1\right)^2}M_{\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta}(x). \end{aligned} \tag{1.2.10}$$

Observe that, the multiple Meixner polynomials of the first kind $M_{\vec{n}}^{\vec{\alpha},\beta}(x)$ are common eigenfunctions of the above two linear difference operators of order $(r+1)$, namely (1.2.9) and (1.2.10).

1.2.3 Multiple Meixner polynomials of the second kind

The monic multiple Meixner polynomial of the second kind [10], corresponding to the multi-index $\vec{n}\in\mathbb{N}^r$ and the set of parameters $\vec{\beta}=(\beta_1,\dots,\beta_r)$, $\beta_i>0$ ($\beta_i-\beta_j\notin\mathbb{Z}$ for all $i\neq j$) and $0<\alpha<1$, is the monic polynomial $M_{\vec{n}}^{\vec{\beta},\alpha}(x)$ of degree $|\vec{n}|$ which satisfies the orthogonality conditions

$$\sum_{x=0}^{\infty} M_{\vec{n}}^{\vec{\beta},\alpha}(x)(-x)_j\upsilon^{\beta_i,\alpha}(x)=0, \qquad j=0,\dots,n_i-1, \qquad i=1,\dots,r,$$

where

$$\upsilon^{\beta_i,\alpha}(x) = \begin{cases} \dfrac{\Gamma(\beta_i+x)}{\Gamma(\beta_i)}\dfrac{\alpha^x}{\Gamma(x+1)}, & \text{if}\quad x\in\mathbb{R}\setminus\left(\mathbb{Z}^-\cup\{-\beta_i,-\beta_i-1,\beta_i-2,\dots\}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Notice that $\upsilon^{\beta_i,\alpha}(x)$ is a C^{∞}-function on $\mathbb{R}\backslash\{-\beta_i,-\beta_i-1,-\beta_i-2,\ldots\}$ with simple poles at the points in $\{-\beta_i,-\beta_i-1,-\beta_i-2,\ldots\}$.

In [10] the normality of the multi-index $\vec{n}\in\mathbb{N}^r$, whenever $\beta_i>0$ and $\beta_i-\beta_j\notin\mathbb{Z}$ when $i\neq j$ was proved. Also the following raising operators were found

$$\mathcal{L}^{\beta_i,\alpha}\left[M_{\vec{n}}^{\vec{\beta},\alpha}(x)\right]=-M_{\vec{n}+\vec{e_i}}^{\vec{\beta}-\vec{e_i},\alpha}(x),\quad i=1,\ldots,r, \tag{1.2.11}$$

where $\mathcal{L}^{\beta_i,\alpha}$ is defined by

$$\mathcal{L}^{\beta_i,\alpha}\overset{\text{def}}{=}\frac{\alpha(\beta_i-1)}{(1-\alpha)\,\upsilon^{\alpha,\beta_i-1}(x)}\bigtriangledown\upsilon^{\alpha,\beta_i}(x).$$

As a consequence of (1.2.11) there holds the Rodrigues-type formula

$$M_{\vec{n}}^{\vec{\beta},\alpha}(x)=\left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\left[\prod_{i=1}^{r}(\beta_i)_{n_i}\right]\frac{\Gamma(x+1)}{\alpha^x}\mathcal{M}_{\vec{n}}^{\vec{\beta}}\left(\frac{\alpha^x}{\Gamma(x+1)}\right), \tag{1.2.12}$$

where

$$\mathcal{M}_{\vec{n}}^{\vec{\beta}}=\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+x)}\bigtriangledown^{n_i}\frac{\Gamma(\beta_i+n_i+x)}{\Gamma(\beta_i+n_i)}.$$

In [65] the author found the high-order linear difference equation for these multiple orthogonal polynomials studied in [10]

$$\prod_{i=1}^{r}\mathcal{L}^{\beta_i+1,\alpha}\left[\triangle M_{\vec{n}}^{\vec{\beta},\alpha}(x)\right]=-\sum_{i=1}^{r}d_i\prod_{\substack{j=1\\j\neq i}}^{r}\mathcal{L}^{\beta_j+1,\alpha}\left[M_{\vec{n}}^{\vec{\beta},\alpha}(x)\right], \tag{1.2.13}$$

where

$$d_i=\frac{\prod\limits_{l=1}^{r}(n_l+\beta_l-\beta_i)}{\prod\limits_{k=1,k\neq i}^{r-1}(\beta_i-\beta_k)\prod\limits_{l=i+1}^{r}(\beta_l-\beta_i)}$$
$$\times\sum_{j=1}^{r}\frac{(-1)^{i+j}\prod\limits_{k=1}^{r}(n_j+\beta_j-\beta_k)}{(n_j+\beta_j-\beta_i)\prod\limits_{k=1,k\neq j}^{r-1}(n_k-n_j+\beta_k-\beta_j)\prod\limits_{l=j+1}^{r}(n_j-n_l+\beta_j-\beta_l)}.$$

Also, in [10, 50] it was found the recurrence relation

$$xM_{\vec{n}}^{\vec{\beta},\alpha}(x)=M_{\vec{n}+\vec{e}_k}^{\vec{\beta},\alpha}(x)+\left[(n_k+\beta_k)\left(\frac{\alpha}{1-\alpha}\right)+\frac{|\vec{n}|}{1-\alpha}\right]M_{\vec{n}}^{\vec{\beta},\alpha}(x)$$
$$+\alpha\sum_{i=1}^{r}\frac{n_i(\beta_i+n_i-1)}{(1-\alpha)^2}\prod_{j\neq i}^{r}\frac{n_i+\beta_i-\beta_j}{n_i-n_j+\beta_i-\beta_j}M_{\vec{n}-\vec{e}_i}^{\vec{\beta},\alpha}(x). \tag{1.2.14}$$

Once more we can see that, the multiple Meixner polynomials of the second kind $M_{\vec{n}}^{\vec{\beta},\alpha}(x)$ are common eigenfunctions of the above two linear difference operators of order $(r+1)$, namely (1.2.13) and (1.2.14).

1.2.4 Multiple Kravchuk polynomials

The multiple monic Kravchuk polynomials [10], corresponding to the multi-indices $\vec{n}\in\mathbb{N}^r$ with $|\vec{n}|\leq N$ and the parameters N and $\vec{p}=(p_1,\ldots,p_r)$, are the unique monic polynomials $K_{\vec{n}}^{\vec{p},N}(x)$ of degree $|\vec{n}|$ which satisfy the conditions

$$\sum_{x=0}^{N} K_{\vec{n}}^{\vec{p},N}(x)(-x)_j\upsilon^{p_i,N}(x)=0, \qquad j=0,\ldots,n_i-1, \qquad i=1,\ldots,r,$$

where

$$\upsilon^{p_i,N}(x)=\begin{cases} \dfrac{N!p_i^x\left(1-p_i\right)^{N-x}}{\Gamma(x+1)\Gamma\left(N-x+1\right)}, & \text{if } x\in\mathbb{R}\backslash\left(\mathbb{Z}^-\cup\{N+1,N+2,\ldots\}\right), \\ 0, & \text{otherwise.}\end{cases}$$

Notice that $\upsilon^{p_i,N}(x)$ is a C^∞-function on $\mathbb{R}$. Furthermore, in [10] the normality of the the multi-index $\vec{n}\in\mathbb{N}^r$ with $|\vec{n}|\leq N$, whenever $0<p_i<1$, $i=1,2,\ldots,r$, and with all the p_i different was proved. The following raising operators were found

$$\mathcal{L}^{p_i,N}\left[K_{\vec{n}}^{\vec{p},N}(x)\right]=-K_{\vec{n}+\vec{e}_i}^{\vec{p},N+1}(x), \quad i=1,\ldots,r, \tag{1.2.15}$$

where $\mathcal{L}^{p_i,N}$ is defined by

$$\mathcal{L}^{p_i,N}\stackrel{\text{def}}{=}\frac{p_i\left(1-p_i\right)\left(N+1\right)}{\upsilon^{p_i,N+1}(x)}\bigtriangledown\upsilon^{p_i,N}(x).$$

As a consequence of (1.2.15) there holds the Rodrigues-type formula

$$K_{\vec{n}}^{\vec{p},N}(x)\;=\;(-N)_{|\vec{n}|}\left(\prod_{i=1}^{r}p_i^{n_i}\right)\frac{\Gamma(x+1)\Gamma(N-x+1)}{N!}\mathcal{K}_{\vec{n}}^{\vec{p}}\frac{(N-|\vec{n}|)!}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}, \tag{1.2.16}$$

where

$$\mathcal{K}_{\vec{n}}^{\vec{p}}=\prod_{i=1}^{r}\left(\frac{1-p_i}{p_i}\right)^x\bigtriangledown^{n_i}\left(\frac{p_i}{1-p_i}\right)^x.$$

In [65] the author found the high-order linear difference equation for these multiple orthogonal polynomials studied in [10]

$$\prod_{i=1}^{r}\mathcal{L}^{p_i,N+r-i-1}\left[\bigtriangleup K_{\vec{n}}^{\vec{p},N}(x)\right]=-\sum_{i=1}^{r}n_i\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{L}^{p_j,N+r-j-1}\left[K_{\vec{n}}^{\vec{p},N}(x)\right]. \tag{1.2.17}$$

Also, in [10, 50] it was found the recurrence relation

$$xK_{\vec{n}}^{\vec{p},N}(x) = K_{\vec{n}+\vec{e}_k}^{\vec{p},N}(x) + \left[(N-|\vec{n}|)p_k + \sum_{i=1}^{r} n_i\,(1-p_i)\right] K_{\vec{n}}^{\vec{p},N}(x)$$
$$+ \sum_{i=1}^{r} n_i p_i\,(p_i-1)\,(|\vec{n}|-N-1)\,K_{\vec{n}-\vec{e}_i}^{\vec{p},N}(x). \quad (1.2.18)$$

The multiple Kravchuk polynomials $K_{\vec{n}}^{\vec{p},N}(x)$ are common eigenfunctions of the above two linear difference operators of order $(r+1)$, namely (1.2.17) and (1.2.18).

1.2.5 Multiple Hahn polynomials

The monic multiple Hahn polynomials [10, 12], corresponding to the multi-indices $\vec{n}\in\mathbb{N}^r$ with $|\vec{n}|\le N$ and the parameters $\vec{\alpha}=(\alpha_1,\dots,\alpha_r)$, α_0 and N, are the unique monic polynomials $H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x)$ of degree $|\vec{n}|$ which satisfy the conditions

$$\sum_{x=0}^{N} H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x)(-x)_j\upsilon^{\alpha_i,\alpha_0,N}(x)=0, \quad j=0,\dots,n_i-1, \quad i=1,\dots,r,$$

where

$$\upsilon^{\alpha_i,\alpha_0,N}(x)=\begin{cases}\dfrac{\Gamma(\alpha_i+x+1)}{\Gamma(x+1)}\dfrac{\Gamma\left(\alpha_0+N-x+1\right)}{\Gamma\left(N-x+1\right)}, & \text{if } x=0,1,\dots,N,\\ \\ 0, & \text{otherwise.}\end{cases}$$

Notice that $\upsilon^{\alpha_i,\alpha_0,N}(x)$ is a C^∞-function on $\mathbb{R}\setminus\left(\{\alpha-1,\alpha-2,\dots\}\cup\{\alpha_0+N+1,\alpha_0+N+2,\dots\}\right)$ with simple poles at the points of $\{\alpha-1,\alpha-2,\dots\}\cup\{\alpha_0+N+1,\alpha_0+N+2,\dots\}$. Furthermore, in [10] the normality of the the multi-index $\vec{n}\in\mathbb{N}^r$ with $|\vec{n}|\le N$, whenever $\alpha_i-\alpha_j\notin\{0,1,\dots,N-1\}$, when $i\neq j$ was proved. In addition, the following raising operators were found

$$\mathcal{L}^{\alpha_i,\alpha_0,N}\left[H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x)\right] = -H_{\vec{n}+\vec{e}_i}^{\vec{\alpha}-\vec{e}_i,\alpha_0-1,N+1}(x), \quad i=1,\dots,r, \quad (1.2.19)$$

where $\mathcal{L}^{\alpha_i,\alpha_0,N}$ is defined by

$$\mathcal{L}^{\alpha_i,\alpha_0,N} \overset{\text{def}}{=} \frac{\left(|\vec{n}|+\alpha_i+\alpha_0\right)^{-1}}{\upsilon^{\alpha_i-1,\alpha_0-1,N+1}(x)}\,\nabla\,\upsilon^{\alpha_i,\alpha_0,N}(x).$$

As a consequence of (1.2.19) there holds the Rodrigues-type formula

$$H_{\vec{n}}^{\vec{\alpha},\alpha_0,N} = \frac{(-1)^{|\vec{n}|}}{\prod_{i=1}^{r}\left(|\vec{n}|+\alpha_i+\alpha_0+1\right)_{n_i}}\frac{\Gamma(x+1)\Gamma(N-x+1)}{\Gamma(\alpha_0+N-x+1)}$$
$$\times\,\mathcal{H}_{\vec{n}}^{\vec{\alpha}}\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}, \quad (1.2.20)$$

where

$$\mathcal{H}_{\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r} \left(\frac{1}{\Gamma(\alpha_i + x + 1)} \bigtriangledown^{n_i} \Gamma(\alpha_i + n_i + x + 1) \right). \tag{1.2.21}$$

In [65] the author finds the high-order linear difference equation for these multiple orthogonal polynomials studied in [10]

$$\begin{aligned} &\prod_{i=1}^{r} \mathcal{L}^{\alpha_i+1,\alpha_0-r+i+1,N+r-i-1} \left[\triangle H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x) \right] \\ &\qquad = -\sum_{i=1}^{r} \left(|\vec{n}| + \alpha_i + \alpha_0 + 1 \right) d_i \prod_{\substack{j=1 \\ j\neq i}}^{r} \mathcal{L}^{\alpha_j+1,\alpha_0-r+i+1,N+r-i-1} \left[H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x) \right], \end{aligned} \tag{1.2.22}$$

where

$$\begin{aligned} d_i = &\frac{\prod_{l=1}^{r} (n_l + \alpha_l - \alpha_i)}{(|\vec{n}| + \alpha_i + \alpha_0 + 1) \prod_{k=1,k\neq i}^{r-1} (\alpha_i - \alpha_k) \prod_{l=i+1}^{r} (\alpha_l - \alpha_i)} \\ &\times \sum_{j=1}^{r} \frac{(-1)^{i+j} (|\vec{n}| + \alpha_j + \alpha_0 + 1) \prod_{k=1}^{r} (n_j + \alpha_j - \alpha_k)}{(n_j + \alpha_j - \alpha_i) \prod_{k=1,k\neq j}^{r-1} (n_k - n_j + \alpha_k - \alpha_j) \prod_{l=j+1}^{r} (n_j - n_l + \alpha_j - \alpha_l)}. \end{aligned}$$

In closing, notice that the multiple Hahn polynomials $H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x)$ are common eigenfunctions of the above linear difference operator of order $(r+1)$, (1.2.22) and recurrence relation (3.3.4).

1.3 q-Multiple orthogonal polynomials

Let us begin by recalling the definition of q-multiple orthogonal polynomials [9].

Definition 1.3.1. A polynomial $P_{\vec{n}}(x(s))$ on the lattice $x(s) = c_1 q^s + c_3$, $q \in \mathbb{R}^+ \setminus \{1\}$, $c_1, c_3 \in \mathbb{R}$, is said to be a q-multiple orthogonal polynomial of a multi-index $\vec{n} \in \mathbb{N}^r$ with respect to positive discrete measures $\mu_1, \mu_2, \ldots, \mu_r$ such that $\operatorname{supp}(\mu_i) \subset \Omega_i \subset \mathbb{R}$, $i = 1, 2, \ldots, r$, if there hold conditions

$$\begin{aligned} &(a) \quad \deg P_{\vec{n}}(x(s)) \leq |\vec{n}| = n_1 + n_2 + \cdots + n_r, \\ &(b) \quad \sum_{s=0}^{N_i} P_{\vec{n}}(x(s)) x(s)^k d\mu_i = 0, \qquad k = 0, \ldots, n_i - 1, \qquad N_i \in \mathbb{N} \cup \{+\infty\}. \end{aligned} \tag{1.3.1}$$

Recall that the q-Gamma function is given by

$$\Gamma_q(s) = \begin{cases} f(s;q) = (1-q)^{1-s}\dfrac{\prod\limits_{k\geq 0}(1-q^{k+1})}{\prod\limits_{k\geq 0}(1-q^{s+k})}, & 0<q<1, \\ q^{\frac{(s-1)(s-2)}{2}} f\big(s;q^{-1}\big), & q>1. \end{cases} \tag{1.3.2}$$

See also [46, 73] for the above definition of the q-Gamma function.

We define the q-analogue of the Stirling polynomials denoted by $[s]_q^{(k)}$, which is a polynomial of degree k in the variable $x(s) = (q^s-1)/(q-1)$, as follows

$$[s]_q^{(k)} = \prod_{j=0}^{k-1} \frac{q^{s-j}-1}{q-1} = x(s)x(s-1)\cdots x(s-k+1) \quad \text{for} \quad k>0, \quad \text{and} \quad [s]_q^{(0)} = 1. \tag{1.3.3}$$

In addition, the following difference operators will appear regularly in the sequel

$$\Delta \overset{\text{def}}{=} \frac{\triangle}{\triangle x(s-1/2)}, \qquad \nabla \overset{\text{def}}{=} \frac{\triangledown}{\triangledown x(s+1/2)}, \tag{1.3.4}$$

$$\nabla^{n_i} = \underbrace{\nabla\cdots\nabla}_{n_i \text{ times}}, \tag{1.3.5}$$

and $\triangledown x_1(s) \overset{\text{def}}{=} \triangledown x(s+1/2) = \triangle x(s-1/2) = q^{s-1/2}$.

1.3.1 q-Hahn multiple orthogonal polynomials

The monic q-Hahn multiple orthogonal polynomial [11], corresponding to the multi-index $\vec{n}\in\mathbb{N}^r$ and the set of parameters N, α_0 and $\vec{\alpha} = (\alpha_1,\dots,\alpha_r)$, is the unique monic polynomial $H_{q,\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)$ of degree $|\vec{n}|$ which satisfies the orthogonality conditions

$$\sum_{s=0}^{N} H_{q,\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)\,[s]_q^{(k)} \upsilon_q^{\alpha_i,\alpha_0,N}(s) = 0, \qquad 0\leq k\leq n_i-1, \qquad i=1,\dots,r,$$

where

$$\upsilon_q^{\alpha_i,\alpha_0,N}(s) = \begin{cases} \dfrac{q^{\frac{\alpha_i+\alpha_0}{2}s}\tilde{\Gamma}_q(s+\alpha_i+1)\tilde{\Gamma}_q(N+\alpha_0-s+1)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s+1)}, & \text{if} \quad s=0,1,\dots,N, \\ 0, & \text{otherwise}, \end{cases}$$

where $\tilde{\Gamma}_q(s) = q^{-\frac{(s-1)(s-2)}{4}}\Gamma_q(s)$ and $\alpha_0,\alpha_i > -1$, $\alpha_i-\alpha_j \notin \{0,1,\dots,N-1\}$ when $i\neq j$. Notice that $\upsilon_q^{\alpha_i,\alpha_0,N}(s)$ is a C^∞-function on $\mathbb{R}\setminus(\{-\alpha_i-1,-\alpha_i-2,\dots\}\cup\{\alpha_0+N+1,\alpha_0+N+2,\dots\})$ with simple poles at the points of $\{-\alpha_i-1,-\alpha_i-2,\dots\}\cup\{\alpha_0+N+1,\alpha_0+N+2,\dots\}$.

In [11] the normality of the multi-index $\vec{n} \in \mathbb{N}^r$, with $|\vec{n}| < N+1$ whenever $\alpha_i > 0$ and $\alpha_i - \alpha_j \notin \{0, 1, \ldots, N-1\}$ when $i \neq j$ was addressed. Moreover, the following raising operators were found

$$\mathcal{D}_q^{\alpha_i,\alpha_0,N}\left[H_{q,\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)\right] = -q^{-\frac{N+|\vec{n}|+\alpha_0}{2}}\left[|\vec{n}|+\alpha_i+\alpha_0\right]_q H_{q,\vec{n}+\vec{e_i}}^{\vec{\alpha}-\vec{e_i},\alpha_0-1,N+1}(s), \quad i = 1, \ldots, r, \quad (1.3.6)$$

where $\mathcal{D}_q^{\alpha_i,\alpha_0,N}$ is defined by

$$\mathcal{D}_q^{\alpha_i,\alpha_0,N} \stackrel{\text{def}}{=} \left(\frac{1}{\upsilon_q^{\alpha_i-1,\alpha 0-1,N+1}(s)}\nabla \upsilon_q^{\alpha_i,\alpha_0,N}(s)\right)$$

and

$$[x]_q \stackrel{\text{def}}{=} \frac{q^{\frac{x}{2}}-q^{-\frac{x}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}.$$

As a consequence of (1.3.6) there holds the q-Rodrigues-type formula

$$H_{q,\vec{n}}^{\vec{\alpha},\alpha_0,N}(s) = \frac{(-1)^{|\vec{n}|}q^{\frac{(N+\alpha_0)|\vec{n}|)+\prod_{i=1}^{r} n_i}{2}+\sum_{i=1}^{r}\binom{n_i}{2}}}{\prod_{k=1}^{r}(|\vec{n}|+\alpha_0+\alpha_k+1|q)_{n_k}}\frac{q^{-\frac{\alpha_0}{2}s}\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-s+1)}{\tilde{\Gamma}_q(\alpha_0+N-s+1)} \\ \times \mathcal{H}_{q,\vec{n}}^{\vec{\alpha}}\frac{q^{\frac{\alpha_0+|\vec{n}|}{2}s}\tilde{\Gamma}_q(\alpha_0+N-s+1)}{\tilde{\Gamma}_q(s+1)\tilde{\Gamma}_q(N-|\vec{n}|-s+1)},$$

where

$$\mathcal{H}_{q,\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r}\mathcal{H}_{q,n_i}^{\alpha_i}, \quad \mathcal{H}_{q,n_i}^{\alpha_i} = \frac{q^{-\frac{\alpha_i}{2}s}}{\tilde{\Gamma}_q(\alpha_i+s+1)}\nabla^{n_i}\frac{\tilde{\Gamma}_q(\alpha_i+n_i+s+1)}{q^{-\frac{\alpha_i+n_i}{2}s}}$$

and $(a|q)_k = \prod_{m=0}^{k-1}[a+m]_q = \tilde{\Gamma}_q(a+k)/\tilde{\Gamma}_q(a)$ that is the q-analogue of the Pochhammer symbol.

In [11] the author finds the high-order linear difference equation for these multiple orthogonal polynomials

$$\prod_{i=1}^{r}\mathcal{D}_q^{\alpha_i+1,\alpha_0+1,N-1}\left[\triangle H_{q,\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)\right] \\ + q^{-\frac{N+|\vec{n}|+\alpha_0-1}{2}}\sum_{i=1}^{r}\xi_i[|\vec{n}|+\alpha_0+\alpha_i+1]_q\mathcal{D}_q^{\alpha_i+1,\alpha_0+1,N-1}\left[H_{q,\vec{n}}^{\vec{\alpha},\alpha_0,N}(s)\right] = 0,$$

where

$$\xi_i = \left(\sum_{k=1}^{r}\frac{(-1)^{k+l}[n_k+\alpha_k+\alpha_0+1]_q\prod_{i=1,i\neq l}^{r}[n_k+\alpha_k-\alpha_i]_q}{\prod_{i=1,i\neq k}^{r-1}[n_i+\alpha_i-n_k-\alpha_k]_q\prod_{j=k+1}^{r}[n_k+\alpha_k-n_j-\alpha_j]_q}\right) \\ \times \frac{q^{\frac{|\vec{n}|-1}{2}}\prod_{k=1}^{r-1}\prod_{l=1}^{r}[\alpha_k-\alpha_l]_q[n_l-n_k+\alpha_l-\alpha_k]_q}{\prod_{k=1}^{r}\prod_{l=1}^{r}[n_j+\alpha_l-\alpha_k]_q}.$$

1.4 On some physical applications

Here we sketch the necessary background for dealing with an interesting physical model involving multiple Meixner polynomials introduced in [70]. Our goal is to pose a similar physical problem (a q-deformed physical model) involving q-multiple Meixner polynomials of the first kind.

1.4.1 One-dimensional harmonic oscillator

Let us start recalling that the Hamiltonian of the particle is given by the expression

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \,,$$

where ω is the angular frequency of the oscillator, m is the particle's mass, $\hat{x}$ is the position operator, and $\hat{p} = -i\hbar\dfrac{\partial}{\partial x}$ is the momentum operator.

The differential operator $\hat{H}$ defines an 'eigenvalue-eigenvector' problem through the time-independent Schrödinger equation,

$$\hat{H}\left|\psi\right\rangle = E\left|\psi\right\rangle \,, \tag{1.4.1}$$

where E denotes a time-independent energy level, or eigenvalue, and the solution $|\psi\rangle$ denotes that level's energy eigenstate (eigenvector function $\psi(x)$).

By solving the second order differential equation (1.4.1) representing the eigenvalue problem in the coordinate basis, for the wave function $< x|\psi >= \psi(x)$, one has a family of solutions

$$\psi_n(x) = \frac{1}{\sqrt{2^n\, n!}} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega x^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \qquad n = 0, 1, 2, \ldots, \tag{1.4.2}$$

where

$$\begin{aligned} H_n(z) &= (-1)^n\, e^{z^2} \frac{d^n}{dz^n}\left(e^{-z^2}\right) \\ &= (2z)^n {}_2F_0\left(\begin{matrix} -n/2 & -(n-1)/2 \\ & \text{—} \end{matrix}\,;\, \frac{-1}{z^2}\right), \end{aligned}$$

represents the Hermite polynomials (the symbol ${}_pF_q$ stands for the usual notation of hypergeometric functions [62, 73]). Moreover, the corresponding energy levels are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) = (2n+1)\frac{\hbar}{2}\omega. \tag{1.4.3}$$

Both the polynomial term of the wave function (1.4.2) and the number $(2n+1)$ in (1.4.3) can be obtained by solving a hypergeometric equation (see [73]). This equation is a particular case of the more general setting studied in Chapter 3.

The ladder operator method (due to Paul Dirac) allows us to extract the energy eigenvalues without solving directly the above differential equation. This method is generalizable to more complicated problems (in quantum field theory), which are not part of this Thesis. Define the operators a and its adjoint $a^\dagger$,

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right),$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right).$$

Hence, one can represent

$$\hat{x} = \sqrt{\frac{\hbar}{2}\frac{1}{m\omega}}(a^\dagger + a), \qquad \hat{p} = i\sqrt{\frac{\hbar}{2}m\omega}(a^\dagger - a)\ .$$

The energy eigenstates $|n\rangle$, when operated on by these ladder operators, give

$$a^\dagger|n\rangle = \sqrt{n+1}\ |\ n+1\rangle,$$
$$a|n\rangle = \sqrt{n}\ |\ n-1\rangle.$$

Observe that $a^\dagger$, essentially, appends a single quantum of energy to the oscillator, while a removes a quantum. These operators are also known as creation and annihilation operators, respectively. Consequently, define a number operator N,

$$N = a^\dagger a, \quad \text{where} \quad N\,|n\rangle = n\,|n\rangle\ .$$

The following commutators can be easily obtained by substituting the canonical commutation relation,

$$[a, a^\dagger] = 1, \qquad [N, a^\dagger] = a^\dagger, \qquad [N, a] = -a,$$

and the Hamilton operator can be expressed as

$$\hat{H} = \left(N + \frac{1}{2}\right)\hbar\omega,$$

where the eigenstate of N is also the eigenstate of energy.

1.4.2 r-dimensional harmonic oscillator

In r dimensions, one has r position coordinates $x_1, ..., x_r$. Corresponding to each position coordinate there is a momentum, which we label $p_1, ..., p_r$. The canonical commutation relations between these operators are

$$[x_i, p_j] = i\hbar\delta_{i,j},$$
$$[x_i, x_j] = 0,$$
$$[p_i, p_j] = 0.$$

Analogously, the Hamiltonian for this system is

$$H = \sum_{i=1}^{r} \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 x_i^2 \right).$$

Observe that the form of this Hamiltonian indicates that the r-dimensional harmonic oscillator is exactly analogous to r independent one-dimensional harmonic oscillators with the same mass and string constant. In this context, the quantities $x_1, ..., x_r$ would refer to the positions of each of the r particles. This observation makes the solution straightforward. For a particular set of quantum numbers $\{n\}$, represented by a multi-index $\vec{n}$, the energy eigenfunctions for the r-dimensional oscillator are expressed in terms of the 1-dimensional eigenfunctions as:

$$\langle \mathbf{x} | \psi_{\vec{n}} \rangle = \prod_{i=1}^{r} \langle x_i \mid \psi_{n_i} \rangle,$$

In the ladder operator method, we define for $i = 1, \ldots, r$

$$a_i = \sqrt{\frac{m\omega}{2\hbar}} \left(x_i + \frac{i}{m\omega} p_i \right),$$
$$a_i^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x_i - \frac{i}{m\omega} p_i \right).$$

Similarly to the one-dimensional case, each of the a_i and $a_i^\dagger$ operators lower and raise the energy by $\hbar\omega$, respectively. The Hamiltonian is

$$H = \hbar\omega \sum_{i=1}^{r} \left(a_i^\dagger \, a_i + \frac{1}{2} \right),$$

and the energy levels of the system are

$$E = \hbar\omega \left[|\vec{n}| + \frac{r}{2} \right], \quad \text{where} \quad |\vec{n}| = n_1 + \cdots + n_r,$$

$n_i = 0, 1, 2, \ldots$ (the energy level in dimension i).

1.4.3 Physical model involving multiple Meixner polynomials

In [70] the authors use the annihilation and creation operators a_i, $a_i^\dagger$, where $i = 1, \ldots, r$, satisfying the commutation relations

$$[a_i, a_j^\dagger] = \delta_{i,j}, \qquad [a_i^\dagger, a_j^\dagger] = [a_i, a_j] = 0, \quad i, j = 1, \ldots, r.$$

The generated Lie algebra is formed by r copies of the Heisenberg-Weyl algebra $W_i = span\{a_i, a_i^\dagger, 1\}$. For a more detailed and technical information about orthogonal polynomials in the Lie algebras see [49] as well as [44] for quantum mechanics and polynomials of a discrete variable.

Furthermore, the normalized simultaneous eigenvectors of the r number operators $N_i = a_i^\dagger a_i$ are denoted by

$$|n_1, n_2, \dots, n_r\rangle = |n_1\rangle|n_2\rangle \cdots |n_r\rangle,$$

Indeed,

$$\begin{aligned} N_i|n_1, n_2, \dots, n_r\rangle &= n_i|n_1, n_2, \dots, n_r\rangle, \\ \langle m_1, m_2, \dots, m_r|n_1, n_2, \dots, n_r\rangle &= \delta_{m_1,n_1} \cdots \delta_{m_r,n_r}. \end{aligned}$$

Moreover,

$$\begin{aligned} a_i^\dagger|n_1, n_2, \dots, n_r\rangle &= \sqrt{n_i+1}|n_1, \dots, n_i+1, \dots, n_r\rangle, \\ a_i|n_1, n_2, \dots, n_r\rangle &= \sqrt{n_i}|n_1, \dots, n_i-1, \dots, n_r\rangle, \end{aligned}$$

The Bargmann realization in terms of coordinates z_i, $i = 1, \dots, r$, in $\mathbb{C}^r$ has

$$\begin{aligned} a_i &= \frac{\partial}{\partial z_i}, \quad a_i^\dagger = z_i, \\ \langle z_1, z_2, \dots, z_r|n_1, n_2, \dots, n_r\rangle &= \frac{z_1^{n_1} \cdots z_r^{n_r}}{\sqrt{n_1! \cdots n_r!}}. \end{aligned}$$

For the studied model in [70], let

$$H_i^{\vec{\alpha},\beta} = a_i + \sum_{k=1}^{r} \frac{N_k}{1-\alpha_k} + \left(\frac{\alpha_i}{1-\alpha_i} + \sum_{j=1}^{r} \frac{\alpha_j}{(1-\alpha_j)^2} a_j^\dagger \right) \left(\sum_{k=1}^{r} N_k + \beta \right),$$

$i = 1, \dots, r$, be the set of non-Hermitian operators defined in the universal enveloping algebra formed by the r copies W_i.

The operators making up the H_i, generate an isomorphic Lie algebra to that of the diffeomorphisms in $\mathbb{C}^r$ spanned by vector fields of the form

$$Z = \sum_{i=1}^{r} f_i(\vec{z}) \frac{\partial}{\partial z_i} + g(\vec{z}), \quad \vec{z} = (z_1, \dots, z_r).$$

Interestingly, the authors pointed out that despite the fact in the coordinate realization where

$$a_i = \frac{1}{\sqrt{2}} \left(x_i + \frac{\partial}{\partial x_i} \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left(x_i - \frac{\partial}{\partial x_i} \right),$$

the operators H_i are third order differential operators, they can be considered as Hamiltonians, which will be simultaneously diagonalised by the multiple Meixner polynomials of the first kind.

Consider the states $|x,\vec{\alpha},\beta\rangle$ defined by means of the combination of states $|n_1,\ldots,n_r\rangle$ as:

$$|x,\vec{\alpha},\beta\rangle = N^r_{x,\vec{\alpha},\beta}\sum_{\vec{n}}\frac{M^{\vec{\alpha},\beta}_{\vec{n}}(x)}{\sqrt{n_1!\cdots n_r!}}|n_1,n_2,\ldots,n_r\rangle, \quad x\in\mathbb{N}.$$

Thus,

$$\begin{aligned}H^{\vec{\alpha},\beta}_i|x,\vec{\alpha},\beta\rangle = N^r_{x,\vec{\alpha},\beta}\sum_{\vec{n}}\frac{1}{\sqrt{n_1!\cdots n_r!}}\Big[M^{\vec{\alpha},\beta}_{\vec{n}+\vec{e}_i}(x)\\
+\left((\beta+|\vec{n}|)\left(\frac{\alpha_i}{1-\alpha_i}\right)+\sum_{k=1}^{r}\frac{n_k}{1-\alpha_k}\right)M^{\vec{\alpha},\beta}_{\vec{n}}(x)\\
+\sum_{j=1}^{r}\frac{\alpha_j n_j\,(\beta+|\vec{n}|-1)}{(\alpha_j-1)^2}M^{\vec{\alpha},\beta}_{\vec{n}-\vec{e}_j}(x)\Big]|n_1,n_2,\ldots,n_r\rangle.\end{aligned}$$

In [70], by using the recurrence relation for multiple Meixner polynomials of the first kind (1.2.10)

$$H^{\vec{\alpha},\beta}_i|x,\vec{\alpha},\beta\rangle = x|x,\vec{\alpha},\beta\rangle,$$

yields. Although operators are non-Hermitian, they have a real spectrum given by the non-negative integers. The states $|x,\vec{\alpha},\beta\rangle$ are uniquely defined as the joint eigenstates of the Hamiltonian operators with eigenvalues equal to x. Moreover,

$$[H^{\vec{\alpha},\beta}_i,H^{\vec{\alpha},\beta}_j]|x,\vec{\alpha},\beta\rangle = 0.$$

However, these Hamiltonians do not commute pairwise. Indeed,

$$[H^{\vec{\alpha},\beta}_i,H^{\vec{\alpha},\beta}_j] = a_i - a_j + \frac{\alpha_i-\alpha_j}{(1-\alpha_i)(1-\alpha_j)}\left(\beta+\sum_{k=1}^{r}N_k\right).$$

Finally, because they do not commute and yet have common eigenvectors, the authors in [70] say that they form a ‘weakly’ integrable system.

The physical model described here will serve as a starting point for a new q-deformed physical model that we will solve by using the results of Chapters 2 and 3 involving the q-multiple Meixner polynomials of the first kind. Some q–analogue of quantum physical models were already studied in one dimension (see [22] for a q-analogue of Bargmann space and [83] for q-deformed coherent states with an explicitly known resolution of unity), but the study of the above ‘weakly’ integrable system in r dimensions is still missing in the literature.

Chapter 2

On some families of q-multiple orthogonal polynomials

In this chapter we set the vector measures that we will study in connection with some families of q-multiple orthogonal polynomials.

2.1 Problem formulation

The present Thesis addresses the following problem, which is divided in three parts: First, introduce some q-multiple orthogonal polynomials that are analogous to the discrete families given in Section 1.2. Aimed to pursue this goal, it will be important to address the AT-property of the involved system of q-discrete measures. Second, find the raising and lowering q-operators as well as Rodrigues-type formula, which provides an explicit expression for the investigated q-multiple orthogonal polynomials. Last, but not least, obtain the recurrence relations as well as the q-difference equations with respect to the independent variable $x(s)$.

2.2 AT-property and definition of q-multiple orthogonal polynomials

• q-Charlier multiple orthogonal polynomials

Let us consider the following r positive discrete measures on $\mathbb{R}^+$,

$$\mu_i = \sum_{s=0}^{\infty} \omega_i(k)\delta(k-s), \qquad \omega_i > 0, \qquad i = 1, 2, \ldots, r. \tag{2.2.1}$$

Here $\omega_i(s) = \upsilon_q^{\alpha_i}(s) \bigtriangleup x(s-1/2)$, which involve the q-analogue of Poisson distributions

$$\upsilon_q^{\alpha_i}(s) = \begin{cases} \dfrac{\alpha_i^s}{\Gamma_q(s+1)}, & \text{if} \quad s \in \mathbb{R}^+ \cup \{0\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha_i > 0$, $i = 1, 2, \ldots, r$, with all the α_i different.

Lemma 2.2.1. *The system of functions*

$$\alpha_1^s, x(s)\alpha_1^s, \ldots, x(s)^{n_1-1}\alpha_1^s, \ldots, \alpha_r^s, x(s)\alpha_r^s, \ldots, x(s)^{n_r-1}\alpha_r^s, \tag{2.2.2}$$

with $\alpha_i > 0$, $i = 1, 2, \ldots, r$, and $(\alpha_i/\alpha_j) \neq q^k$, $k \in \mathbb{Z}$, $i, j = 1, \ldots, r$, $i \neq j$, forms a Chebyshev system on $\mathbb{R}^+$ for every $\vec{n} = (n_1, \ldots, n_r) \in \mathbb{N}^r$.

Proof. This means that every linear combination $\sum_{i=1}^{r} Q_{n_i-1}(x(s))\alpha_i^s$ has at most $|\vec{n}| - 1$ zeros on $\mathbb{R}^+$ for every $Q_{n_i-1}(x(s)) \in \mathbb{P}_{n_i-1} \setminus \{0\}$. Since $x(s) = c_1 q^s + c_3$, where c_1, c_3 are constants, we consider $\sum_{i=1}^{r} Q_{n_i-1}(q^s)\alpha_i^s$, instead. Thus, the system (2.2.2) transforms into

$$a_{1,0}^s, a_{1,1}^s, \ldots, a_{1,n_1-1}^s, \ldots, a_{r,0}^s, a_{r,1}^s, \ldots, a_{r,n_r-1}^s,$$

where $a_{i,k} = (q^k \alpha_i)$, with $k = 0, \ldots, n_i - 1$, $i = 1, \ldots, r$. Observe that $a_{j,m} \neq a_{l,p}$ for $j \neq l$, $m \neq p$. Hence, identity $a_{i,k} = e^{\log a_{i,k}}$ yields the well-known Chebyshev system (see [77, p. 138])

$$e^{s \log a_{1,0}}, e^{s \log a_{1,1}}, \ldots, e^{s \log a_{1,n_1-1}}, \ldots, e^{s \log a_{r,0}}, e^{s \log a_{r,1}}, \ldots, e^{s \log a_{r,n_r-1}}.$$

Then, we conclude that the functions (2.2.2) form a Chebyshev system on $\mathbb{R}^+$. □

As a consequence of Lemma 2.2.1 the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (2.2.1) forms an AT system on $\mathbb{R}^+$.

• q-Meixner multiple orthogonal polynomials of the first kind

Let us consider the following r positive discrete measures on $\mathbb{R}^+$,

$$\mu_i = \sum_{s=0}^{\infty} \omega_i(k)\delta(k-s), \qquad \omega_i > 0, \qquad i = 1, 2, \ldots, r. \tag{2.2.3}$$

Here $\omega_i(s) = \upsilon_q^{\alpha_i,\beta}(s) \bigtriangleup x(s-1/2)$, and

$$\upsilon_q^{\alpha_i,\beta}(s) = \begin{cases} \dfrac{\alpha_i^s}{\Gamma_q(s+1)} \dfrac{\Gamma_q(\beta+s)}{\Gamma_q(\beta)}, & \text{if} \quad s \in \mathbb{R}^+ \cup \{0\}, \\ 0, & \text{otherwise}, \end{cases}$$

where $0 < \alpha_i < 1$, $\beta > 0$, $i = 1, 2, \ldots, r$, and with all the α_i different.

As a consequence of Lemma 2.2.2 the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (2.2.3) forms an AT system on $\mathbb{R}^+$.

• **q-Meixner multiple orthogonal polynomials of the second kind**

Let us consider the following r positive discrete measures on $\mathbb{R}^+$,

$$\mu_i = \sum_{s \geq 0} \omega_i (k) \delta (k - s), \quad \omega_i > 0, \quad i = 1, 2, \ldots, r. \tag{2.2.4}$$

Here $\omega_i (s) = \upsilon_q^{\beta_i, \alpha} (s) \triangle x (s - 1/2)$, and

$$\upsilon_q^{\beta_i, \alpha} (s) = \begin{cases} \dfrac{\Gamma_q(\beta_i + s)}{\Gamma_q(\beta_i)} \dfrac{\alpha^x}{\Gamma_q(s+1)}, & \text{if } s \in \Omega, \\ \\ 0, & \text{otherwise,} \end{cases}$$

being $\Omega = \mathbb{R} \setminus \{\mathbb{Z}^- \cup \{-\beta_i, -\beta_i - 1, -\beta_i - 2, \ldots\}\}$, $\beta_i > 0$, $(\beta_i - \beta_j \notin \mathbb{Z} \quad \text{for all} \quad i \neq j)$, $i = 1, 2, \ldots, r$, and $0 < \alpha < 1$.

Lemma 2.2.2. *Let $\upsilon (s)$ be a continuous function with no zeros on $\mathbb{R}^+$, then the functions*

$$\begin{gathered} \upsilon (s) \Gamma_q (s + \beta_1), \upsilon (s) x (s) \Gamma_q (s + \beta_1), \ldots, \upsilon (s) x (s)^{n_1 - 1} \Gamma_q (s + \beta_1), \\ \vdots \\ \upsilon (s) \Gamma_q (s + \beta_r), \upsilon (s) x (s) \Gamma_q (s + \beta_r), \ldots, \upsilon (s) x (s)^{n_r - 1} \Gamma_q (s + \beta_r), \end{gathered} \tag{2.2.5}$$

with $\beta_i > 0$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, form a Chebyshev system on Ω for every $\vec{n} \in \mathbb{N}^r$.

Proof. For the system of functions (2.2.5) with $\beta_i > 0$, $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, and $\upsilon (s)$ a continuous function with no zeros on $\mathbb{R}^+$, we have a Chebyshev system on Ω for every $\vec{n} \in \mathbb{N}^r$ if and only if every linear combination of these functions (except the one with each coefficient equal to 0) has at most $|\vec{n}| - 1$ zeros. This linear combination can be rewriting as a function of the system

$$\begin{aligned} &\upsilon (s) \Gamma_q (s + \beta_1), \upsilon (s) [s + \beta_1]_q^{(1)} \Gamma_q (s + \beta_1), \ldots, \\ &\qquad \upsilon (s) [s + \beta_1 + n_1 - 2]_q^{(n_1 - 1)} \Gamma_q (s + \beta_1), \\ &\qquad \upsilon (s) \Gamma_q (s + \beta_r), \upsilon (s) [s + \beta_r]_q^{(1)} \Gamma_q (s + \beta_r), \ldots, \\ &\qquad \qquad \upsilon (s) [s + \beta_1 + n_r - 2]_q^{(n_r - 1)} \Gamma_q (s + \beta_r), \end{aligned}$$

where $[s + \beta_i]_q^{(n_i)}, \quad i = 1, \ldots, r$, is given in (1.3.3).

Observe that

$$[s + k - 1]_q^{(k)} \Gamma_q (s) = \Gamma_q (s + k),$$

holds. Therefore, the above system transforms into

$$\begin{array}{c} \upsilon\left(s\right)\Gamma_{q}\left(s+\beta_{1}\right), \upsilon\left(s\right)\Gamma_{q}\left(s+\beta_{1}+1\right), \ldots, \upsilon\left(s\right)\Gamma_{q}\left(s+\beta_{1}+n_{1}-1\right), \\ \vdots \\ \upsilon\left(s\right)\Gamma_{q}\left(s+\beta_{r}\right), \upsilon\left(s\right)\Gamma_{q}\left(s+\beta_{r}+1\right), \ldots, \upsilon\left(s\right)\Gamma_{q}\left(s+\beta_{r}+n_{r}-1\right). \end{array} \tag{2.2.6}$$

Thus, it is sufficient to prove that these systems (2.2.6) form a Chebyshev system on Ω for every $\vec{n} \in \mathbb{N}^r$. Indeed, if we define the matrix $\mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right)$ as

$$\begin{pmatrix} \Gamma_q\left(s_1+\beta_1\right) & \Gamma_q\left(s_2+\beta_1\right) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_1\right) \\ \vdots & \vdots & & \vdots \\ \Gamma_q\left(s_1+\beta_1+n_1-1\right) & \Gamma_q\left(s_2+\beta_1+n_1-1\right) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_1+n_1-1\right) \\ \vdots & \vdots & & \vdots \\ \Gamma_q\left(s_1+\beta_r\right) & \Gamma_q\left(s_2+\beta_r\right) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_r\right) \\ \vdots & \vdots & & \vdots \\ \Gamma_q\left(s_1+\beta_r+n_1-1\right) & \Gamma_q\left(s_2+\beta_r+n_1-1\right) & \cdots & \Gamma_q\left(s_{|\vec{n}|}+\beta_r+n_1-1\right) \end{pmatrix},$$

our proof is reduced to show that $\det\mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right) \neq 0$, for every $|\vec{n}|$, and different points $s_1, \ldots, s_{|\vec{n}|}$ in Ω, since $|\upsilon| > 0$ on Ω. We recall the integral representation from the q-gamma function

$$\Gamma_q\left(s\right) = \int_0^{\frac{1}{1-q}} t^{s-1} E_q^{-qt} d_q t = \int_0^{x(\infty)} t^{s-1} E_q^{-qt} d_q t, \quad s > 0, \tag{2.2.7}$$

where

$$E_q^z = {}_0\varphi_0\left(-;-;q,-\left(1-q\right)z\right)$$

is a q-analogue of the exponential function. For details, see, for instance [63]. Now we replace the q-gamma function in $\mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right)$ by (2.2.7). Then, by the multilinearity of the determinant we take $|\vec{n}|$ integrations out of $|\vec{n}|$ rows to obtain

$$\det\mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right) = \underbrace{\int_0^{x(\infty)} \cdots \int_0^{x(\infty)}}_{|\vec{n}| \text{ times}} \prod_{1 \le i \le |\vec{n}|} E_q^{-qt_i} t_i^{s_i-1} \times \det\mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) d_q t_1 \ldots d_q t_{|\vec{n}|}, \tag{2.2.8}$$

where

$$\mathcal{B}\left(\vec{n}, t_1, \ldots, t_{|\vec{n}|}\right) = \begin{pmatrix} t_1^{\beta_1} & t_2^{\beta_1} & \cdots & t_{|\vec{n}|}^{\beta_1} \\ \vdots & \vdots & & \vdots \\ t_1^{\beta_1+n_1-1} & t_2^{\beta_1+n_1-1} & \cdots & t_{|\vec{n}|}^{\beta_1+n_1-1} \\ \vdots & \vdots & & \vdots \\ t_1^{\beta_r} & t_2^{\beta_r} & \cdots & t_{|\vec{n}|}^{\beta_r} \\ \vdots & \vdots & & \vdots \\ t_1^{\beta_r+n_r-1} & t_2^{\beta_r+n_r-1} & \cdots & t_{|\vec{n}|}^{\beta_r+n_r-1} \end{pmatrix}.$$

Notice that, from ([77] p. 138, Example 4) we know that the functions

$$t^{\beta_1},\ldots,t^{\beta_1+n_1-1},\ldots,t^{\beta_r},\ldots,t^{\beta_r+n_r-1},$$

form a Chebyshev system on $\mathbb{R}^+$ if all the exponents are different, which is in accordance with our choice $\beta_i-\beta_j\notin\mathbb{Z}$ whenever $i\neq j$. Moreover, if all $n_i<N+1$, then the exponents involved in the above matrix are different for $\beta_i-\beta_j\notin\{0,1,\ldots,N\}$ whenever $i\neq j$. Hence, $\det\mathcal{B}\left(\vec{n},t_1,\ldots,t_{|\vec{n}|}\right)$ does not vanish for distinct $t_1,\ldots,t_{|\vec{n}|}$. Now, for a permutation σ of $\{1,\ldots,|\vec{n}|\}$ we make a change of variables $t_i\mapsto t_{\sigma(i)}$ in the integral (2.2.8). Thus, we have

$$\det\mathcal{A}\left(\vec{n},t_1,\ldots,t_{|\vec{n}|}\right)=\underbrace{\int_0^{x(\infty)}\cdots\int_0^{x(\infty)}}_{|\vec{n}|\text{ times}}\prod_{1\leq i\leq|\vec{n}|}E_q^{-qt_i}\det\mathcal{B}\left(\vec{n},t_1,\ldots,t_{|\vec{n}|}\right)\times\operatorname{sgn}(\sigma)\prod_{1\leq j\leq|\vec{n}|}t_{\sigma(j)}^{s_j-1}d_qt_1\ldots d_qt_{|\vec{n}|}.\quad(2.2.9)$$

We average (2.2.9) over all permutation σ we get

$$\det\mathcal{A}\left(\vec{n},s_1,\ldots,s_{|\vec{n}|}\right)=\frac{1}{n!}\sum_{\sigma\in S_{|\vec{n}|}}\underbrace{\int_0^{x(\infty)}\cdots\int_0^{x(\infty)}}_{|\vec{n}|\text{ times}}\prod_{1\leq i\leq|\vec{n}|}E_q^{-qt_i}\times\det\mathcal{B}\left(\vec{n},t_1,\ldots,t_{|\vec{n}|}\right)\operatorname{sgn}(\sigma)\prod_{1\leq j\leq|\vec{n}|}t_{\sigma(j)}^{s_j-1}d_qt_1\ldots d_qt_{|\vec{n}|},$$

being $S_{|\vec{n}|}$ the permutation group. Now relabeling the choice of points, i.e., $t_1,\ldots,t_{|\vec{n}|}$, where $0<t_1<\cdots<t_{|\vec{n}|}$, we have

$$\det\mathcal{A}\left(\vec{n},t_1,\ldots,t_{|\vec{n}|}\right)=\frac{1}{n!}\underbrace{\int_0^{x(\infty)}\cdots\int_0^{x(\infty)}}_{0<t_1<\cdots<t_{|\vec{n}|}}\prod_{1\leq i\leq|\vec{n}|}E_q^{-qt_i}\det\mathcal{B}\left(\vec{n},t_1,\ldots,t_{|\vec{n}|}\right)\times\sum_{\sigma\in S_{|\vec{n}|}}\operatorname{sgn}(\sigma)\prod_{1\leq j\leq|\vec{n}|}t_{\sigma(j)}^{s_j-1}d_qt_1\ldots d_qt_{|\vec{n}|}.\quad(2.2.10)$$

As a result, from the definition of determinant we have

$$\sum_{\sigma\in S_{|\vec{n}|}}\operatorname{sgn}(\sigma)\prod_{1\leq j\leq|\vec{n}|}t_{\sigma(j)}^{s_j-1}=\begin{vmatrix}t_1^{s_1-1}&t_1^{s_2-1}&\cdots&t_1^{s_{|\vec{n}|}-1}\\t_2^{s_1-1}&t_2^{s_2-1}&\cdots&t_2^{s_{|\vec{n}|}-1}\\\vdots&\vdots&&\vdots\\t_{|\vec{n}|}^{s_1-1}&t_{|\vec{n}|}^{s_2-1}&\cdots&t_{|\vec{n}|}^{s_{|\vec{n}|}-1}\end{vmatrix}.\quad(2.2.11)$$

Taking into account that $t_1, \ldots, t_{|\vec{n}|}$ are strictly positive and different, and using [77, p. 138, Example 3] with multi-index $(1, \ldots, 1)$, then it implies that (2.2.11) is different from zero if all the $s_1, \ldots, s_{|\vec{n}|}$ are different. Accordingly, being $s_1, \ldots, s_{|\vec{n}|}$ all different, the integrand of equation (2.2.10) has constant sign in the integration area and hence $\det \mathcal{A}\left(\vec{n}, s_1, \ldots, s_{|\vec{n}|}\right)$ is different from zero. □

As a consequence of Lemma 2.2.2 the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (2.2.4) forms an AT system on Ω.

• **q-Kravchuk multiple orthogonal polynomials**

Let us consider the following r positive discrete measures on $\mathbb{R}^+$,

$$\mu_i = \sum_{s=0}^{N} \omega_i(k)\delta(k-s), \qquad \omega_i > 0, \qquad i = 1, 2, \ldots, r. \tag{2.2.12}$$

Here $\omega_i(s) = \upsilon_q^{p_i,N}(s) \bigtriangleup x(s-1/2)$, and

$$\upsilon_q^{p_i,N}(s) = \begin{cases} \dfrac{q^{\binom{s}{2}} [N]_q! p_i^s (1-p_i)^{N-s}}{\Gamma_q(s+1)\Gamma_q(N-s+1)}, & \text{if } s \in \Theta, \\ \\ 0, & \text{otherwise,} \end{cases}$$

being $\Theta = \mathbb{R}\backslash \left(\mathbb{Z}^- \cup \{N+1, N+2, \ldots\}\right)$, $|\vec{n}| \le N$, $0 < p_i < 1$, $i = 1, 2, \ldots, r$, and with all the p_i different.

As a consequence of Lemma 2.2.2 with $\alpha_i = \left(\frac{p_i}{1-p_i}\right)$, the system of measures $\mu_1, \mu_2, \ldots, \mu_r$ given in (2.2.12) forms an AT system on Θ.

2.3 q-Rodrigues-type formula

In this section we define the q-multiple orthogonal polynomials corresponding to the q-vector measures introduced in the previous section. In addition, the raising operators as well as the q-Rodrigues type formula will be obtained.

2.3.1 q-Charlier multiple orthogonal polynomials

Definition 2.3.1. A polynomial $C_{q,\vec{n}}^{\vec{\alpha}}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(k)} \upsilon_q^{\alpha_i}(s) \bigtriangleup x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r, \tag{2.3.1}$$

(see (1.3.1) with respect to the measures (2.2.1)) is said to be the q-Charlier multiple orthogonal polynomial.

Let us point out some observations derived from the above definition. When $r = 1$ we recover the scalar q-Charlier polynomials [3]. The orthogonality conditions (1.3.1) have been written more conveniently as (2.3.1) since $[s]_q^{(k)} = q^{-\binom{k}{2}} x^k(s) + \text{lower terms} = \mathcal{O}(q^{ks})$. Here, the symbol $\mathcal{O}(\cdot)$ stands for big-O notation. Indeed, when q goes to 1, the symbol $[s]_q^{(k)}$ converges to $(-1)^k(-s)_k$, where $(s)_k$ denotes the Pochhammer symbol. Hence, one can recover the multiple Charlier polynomials given in [10] as a limiting case, provided that the lattice $x(s) = (q^s - 1)/(q - 1)$. Moreover, in the sequel we will only use this lattice. Finally, we have an AT-system of positive discrete measures, then the q-Charlier multiple orthogonal polynomial $C_{q,\vec{n}}^{\vec{\alpha}}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^+$ (see [10, Theorem 2.1, pp. 26–27]).

Observe that, by using summation by parts

$$\sum_{k=M}^{N} u(k)\Delta \upsilon(k) = u(N+1)\upsilon(N+1) - u(M)\upsilon(M) - \sum_{k=M}^{N} \upsilon(k+1)\Delta u(k)$$

with $\upsilon_q^{\alpha_i}(-1) = \upsilon_q^{\alpha_i}(\infty) = 0$, we have for any polynomials ϕ and ψ,

$$\sum_{s\geqslant 0} \Delta\phi(s)\psi(s)\upsilon_q^{\alpha_i}(s) \bigtriangledown x_1(s) = -\sum_{s\geqslant 0} \phi(s)\nabla\left[\psi(s)\upsilon_q^{\alpha_i}(s)\right] \bigtriangleup x(s-1/2). \tag{2.3.2}$$

Then, replacing $[s]_q^{(k)}$ in (2.3.1) by the following finite-difference expression

$$[s]_q^{(k)} = \frac{q^{k-1/2}}{[k+1]_q^{(1)}} \nabla [s+1]_q^{(k+1)}, \tag{2.3.3}$$

one has

$$\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)\nabla[s+1]_q^{(k+1)}\upsilon_q^{\alpha_i}(s) \bigtriangleup x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r.$$

Remark: Observe that the canonical basis in the space of q-polynomials in the q-linear lattice $x(s)$ is $[s]_q^{(k)}$, $k = 0, 1, \ldots$.

Now, by using (2.3.2) one gets

$$\begin{aligned}\sum_{s=0}^{\infty} \nabla\left[C_{q,\vec{n}}^{\vec{\alpha}}(s)\upsilon_q^{\alpha_i}(s)\right] [s]_q^{(k+1)} \bigtriangleup x(s-1/2) &= -\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)\upsilon_q^{\alpha_i}(s)\Delta[s]_q^{(k+1)} \bigtriangleup x(s-1/2) \\ &= -\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)\upsilon_q^{\alpha_i}(s)\nabla[s+1]_q^{(k+1)} \bigtriangleup x(s-1/2).\end{aligned}$$

Thus

$$\sum_{s=0}^{\infty} \nabla\left[C_{q,\vec{n}}^{\vec{\alpha}}(s)\upsilon_q^{\alpha_i}(s)\right] [s]_q^{(k+1)} \bigtriangleup x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r.$$

On the other hand we have

$$\begin{aligned}\nabla\left[C^{\vec{\alpha}}_{q,\vec{n}}(s)\upsilon^{\alpha_i}_q(s)\right] &= -\alpha_i^{-1}q^{-|\vec{n}|}q^{1/2}\upsilon^{\alpha_i/q}_q(s)\left(x^{|\vec{n}|+1}+\cdots\right)\\ &= -\alpha_i^{-1}q^{-|\vec{n}|}q^{1/2}\upsilon^{\alpha_i/q}_q(s)\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s).\end{aligned}$$

Consequently

$$\sum_{s=0}^{\infty}\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s)\upsilon^{\alpha_i/q}_q(s)[s]^{(k+1)}_q \bigtriangleup x(s-1/2)=\sum_{s=0}^{\infty}\nabla\left[C^{\vec{\alpha}}_{q,\vec{n}}(s)\upsilon^{\alpha_i}_q(s)\right][s]^{(k+1)}_q \bigtriangleup x(s-1/2)=0.$$

Then from (2.3.1) we deduce that $\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s) = C^{\vec{\alpha}_{i,1/q}}_{q,\vec{n}+\vec{e}_i}(s)$ where $\vec{\alpha}_{i,1/q} = (\alpha_1,\ldots,\alpha_i/q,\ldots,\alpha_r)$. Therefore

$$\nabla\left[C^{\vec{\alpha}}_{q,\vec{n}}(s)\upsilon^{\alpha_i}_q(s)\right] = -\alpha_i^{-1}q^{-|\vec{n}|}q^{1/2}\upsilon^{\alpha_i/q}_q(s)C^{\vec{\alpha}_{i,1/q}}_{q,\vec{n}+\vec{e}_i}(s).$$

Thus, for monic q-Charlier multiple orthogonal polynomials we have r raising operators

$$\mathcal{D}^{\alpha_i}_q C^{\vec{\alpha}}_{q,\vec{n}}(s) = -q^{1/2}C^{\vec{\alpha}_{i,1/q}}_{q,\vec{n}+\vec{e}_i}(s), \qquad i=1,\ldots,r, \tag{2.3.4}$$

where

$$\mathcal{D}^{\alpha_i}_q \overset{\text{def}}{=} \left(\frac{\alpha_i q^{|\vec{n}|}}{\upsilon^{\alpha_i/q}_q(s)}\nabla\upsilon^{\alpha_i}_q(s)\right), \qquad \vec{\alpha}_{i,1/q} = (\alpha_1,\ldots,\alpha_i/q,\ldots,\alpha_r).$$

Notice that we call $\mathcal{D}^{\alpha_i}_q$ a raising operator since the i-th component of the multi-index $\vec{n}$ in (2.3.4) is increased by 1.
Furthermore,

$$q^{-|\vec{n}|-1/2}\mathcal{D}^{\alpha_i}_q f(s) = [\alpha_i - x(s)]f(s) + x(s)\bigtriangledown f(s),$$

for any function $f(s)$ defined on the discrete variable s.

In sections 3.1.1 and 3.2.1 we will only consider monic q-Charlier multiple orthogonal polynomials.

Proposition 2.3.2. *There holds the following q-analogue of Rodrigues-type formula*

$$C^{\vec{\alpha}}_{q,\vec{n}}(s) = \mathcal{G}^{\vec{n},\vec{\alpha}}_q\Gamma_q(s+1)\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}\left(\frac{1}{\Gamma_q(s+1)}\right), \tag{2.3.5}$$

where

$$\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}} = \prod_{i=1}^{r}\mathcal{C}^{\alpha_i}_{q,n_i}, \quad \mathcal{C}^{\alpha_i}_{q,n_i} = (\alpha_i)^{-s}\nabla^{n_i}(\alpha_i q^{n_i})^s \tag{2.3.6}$$

and

$$\mathcal{G}^{\vec{n},\vec{\alpha}}_q = (-1)^{|\vec{n}|}q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\alpha_i^{n_i}\right)\left(\prod_{i=1}^{r}q^{n_i\sum\limits_{j=i}^{r}n_j}\right).$$

Proof. Applying the raising operators (2.3.4) in a recursive way one obtains the Rodrigues-type formula (2.3.5).

Observe that, for $i=1$ applying (2.3.4) k_1-times we get

$$\left(\frac{\alpha_1}{q^{k_1}}\right)^{-s}\nabla^{k_1}\frac{\alpha_1^s}{\Gamma_q(s+1)}C_{q,n_1}^{\alpha_1}(s)=(-1)^{k_1}q^{k_1/2}q^{-k_1n_1}\alpha_1^{-k_1}C_{q,n_1}^{\alpha_1/q^{k_1}}(s)\frac{1}{\Gamma_q(s+1)}.$$

For $i=2$ in k_2-times we get

$$\begin{aligned}&\left(\frac{\alpha_2}{q^{k_2}}\right)^{-s}\nabla^{k_2}\alpha_2^s\left(\frac{\alpha_1}{q^{k_1}}\right)^{-s}\nabla^{k_1}\frac{\alpha_1^s}{\Gamma_q(s+1)}C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s)\\&\qquad=(-1)^{k_1+k_2}q^{(k_1+k_2)/2}q^{-k_1n_1}q^{-k_2(n_1+k_1+n_2)}\alpha_1^{-k_1}\alpha_2^{-k_2}C_{q,n_1+k_1,n_2+k_2}^{\alpha_1/q^{k_1},\alpha_2/q^{k_2}}(s)\frac{1}{\Gamma_q(s+1)}.\end{aligned}$$

Thus, by induction we get

$$\begin{aligned}&\prod_{i=1}^{r}\left(\frac{\alpha_i}{q^{k_i}}\right)^{-s}\nabla^{k_i}\alpha_i^s\frac{1}{\Gamma_q(s+1)}C_{q,\vec{n}}^{\vec{\alpha}}(s)\\&\qquad=(-1)^{|\vec{k}|}q^{|\vec{k}|/2}\prod_{i=1}^{r}\alpha_i^{-k_i}\prod_{i=1}^{r}q^{-n_i\sum\limits_{j=i}^{r}k_j}\prod_{i=1}^{r-1}q^{-k_i\sum\limits_{j=i+1}^{r}k_j}C_{q,\vec{n}+\vec{k}}^{\alpha_1/q^{k_1},\dots,\alpha_r/q^{k_r}}(s)\frac{1}{\Gamma_q(s+1)},\end{aligned}$$

$\vec{k}=(k_1,\dots,k_r)\in\mathbb{N}^r$ and $|\vec{k}|=k_1+\cdots+k_r$.
Taking $n_1=n_2=\cdots=n_r=0$, $C_{q,\vec{0}}^{\vec{\alpha}}(s)=1$, and $\alpha_i=\alpha_iq^{k_i}$ we get

$$C_{q,\vec{n}+\vec{k}}^{\alpha_1/q^{k_1},\dots,\alpha_r/q^{k_r}}(s)=(-1)^{|\vec{k}|}q^{-|\vec{k}|/2}\prod_{i=1}^{r}\alpha_i^{k_i}\prod_{i=1}^{r}q^{k_i\sum\limits_{j=i}^{r}k_j}\prod_{i=1}^{r}(\alpha_i)^{-s}\nabla^{k_i}\left(\alpha_iq^{k_i}\right)^s\frac{1}{\Gamma_q(s+1)}.$$

Finally, taking $k_i=n_i$, $i=1,\dots,r$, in the above expression, (2.3.5) holds. □

2.3.2 q-Meixner multiple orthogonal polynomials of the first kind

Definition 2.3.3. A polynomial $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$, with multi-index $\vec{n}\in\mathbb{N}^r$ and degree $|\vec{n}|$, that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty}M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(k)}\upsilon_q^{\alpha_i,\beta}(s)\triangle x(s-1/2)=0,\qquad 0\le k\le n_i-1,\qquad i=1,\dots,r,\tag{2.3.7}$$

(see (1.3.1) with respect to the measures (2.2.3)) is said to be the q-Meixner multiple orthogonal polynomial of the first kind.

The orthogonality conditions (1.3.1) have been written more conveniently as (2.3.7). Here, one can recover the multiple Meixner polynomials of the first kind given in [10] as a limiting case. Moreover, we have an AT-system of positive discrete measures, as a result the q-Meixner multiple orthogonal polynomial of the first kind $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^+$ (see [10, Theorem 2.1, pp. 26–27]).

Notice that, replacing $[s]_q^{(k)}$ in (2.3.7) by (2.3.3) one has

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\nabla[s+1]_q^{(k+1)}\upsilon_q^{\alpha_i,\beta}(s)\bigtriangleup x(s-1/2)=0, \qquad 0\leq k\leq n_i-1, \qquad i=1,\ldots,r.$$

Using (2.3.2), one gets

$$\begin{aligned}\sum_{s=0}^{\infty}\nabla\left[M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\upsilon_q^{\alpha_i,\beta}(s)\right][s]_q^{(k+1)}\bigtriangleup x(s-1/2) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\upsilon_q^{\alpha_i,\beta}(s)\Delta[s]_q^{(k+1)}\bigtriangleup x(s-1/2)\\ &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\upsilon_q^{\alpha_i,\beta}(s)\nabla[s+1]_q^{(k+1)}\bigtriangleup x(s-1/2).\end{aligned}$$

Thus

$$\sum_{s=0}^{\infty}\nabla\left[M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\upsilon_q^{\alpha_i,\beta}(s)\right][s]_q^{(k+1)}\bigtriangleup x(s-1/2)=0, \qquad 0\leq k\leq n_i-1, \qquad i=1,\ldots,r.$$

On the other hand we have

$$\begin{aligned}\nabla\left[M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\upsilon_q^{\alpha_i,\beta}(s)\right] &= -\frac{\alpha_i^{-1}q^{-|\vec{n}|}q^{1/2}\left(1-\alpha_i q^{|\vec{n}|+\beta-1}\right)}{x(\beta-1)}\upsilon_q^{\alpha_i/q,\beta-1}(s)\left(x^{|\vec{n}|+1}+\cdots\right)\\ &= -\frac{\alpha_i^{-1}q^{-|\vec{n}|}q^{1/2}\left(1-\alpha_i q^{|\vec{n}|+\beta-1}\right)}{x(\beta-1)}\upsilon_q^{\alpha_i/q,\beta-1}(s)\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s).\end{aligned}$$

Consequently

$$\sum_{s=0}^{\infty}\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s)\upsilon_q^{\alpha_i/q,\beta-1}(s)[s]_q^{(k+1)}\bigtriangleup x(s-1/2)=\sum_{s=0}^{\infty}\nabla\left[M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\upsilon_q^{\alpha_i,\beta}(s)\right][s]_q^{(k+1)}\bigtriangleup x(s-1/2)=0.$$

Then from (2.3.7) we deduce that $\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s)=M_{q,\vec{n}+\vec{e}_i}^{\vec{\alpha}_{i,1/q},\beta-1}(s)$. Therefore

$$\nabla\left[M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\upsilon_q^{\alpha_i,\beta}(s)\right] = -\frac{\alpha_i^{-1}q^{-|\vec{n}|}q^{1/2}\left(1-\alpha_i q^{|\vec{n}|+\beta-1}\right)}{x(\beta-1)}\upsilon_q^{\alpha_i/q,\beta-1}(s)M_{q,\vec{n}+\vec{e}_i}^{\vec{\alpha}_{i,1/q},\beta-1}(s).$$

Then, for monic q-Meixner multiple orthogonal polynomials of the first kind we have r raising operators

$$\mathcal{D}_q^{\alpha_i,\beta}M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)=-q^{1/2}M_{q,\vec{n}+\vec{e}_i}^{\vec{\alpha}_{i,1/q},\beta-1}(s), \qquad i=1,\ldots,r, \tag{2.3.8}$$

where

$$\mathcal{D}_q^{\alpha_i,\beta} \stackrel{\text{def}}{=} \left(\frac{\alpha_i x\,(\beta-1)\,q^{|\vec{n}|}}{\left(1-\alpha_i q^{|\vec{n}|+\beta-1}\right)\upsilon_q^{\alpha_i/q,\beta-1}(s)} \nabla \upsilon_q^{\alpha_i,\beta}(s) \right).$$

Furthermore,

$$q^{-|\vec{n}|-1/2}\mathcal{D}_q^{\alpha_i,\beta} f(s) = \frac{1}{1-\alpha_i q^{|\vec{n}|+\beta-1}} \left[\alpha_i q^{\beta-1}\left(x(s)-x(1-\beta)\right) - x(s)\right] f(s) + \frac{1}{1-\alpha_i q^{|\vec{n}|+\beta-1}} x(s) \bigtriangledown f(s),$$

for any function $f(s)$ defined on the discrete variable s. We call $\mathcal{D}_q^{\alpha_i,\beta}$ a raising operator since the i-th component of the multi-index $\vec{n}$ in (2.3.8) is increased by 1.

In the sequel we will only consider monic q-Meixner multiple orthogonal polynomials of the first kind.

Proposition 2.3.4. *There holds the following q-analogue of Rodrigues-type formula*

$$M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \mathcal{G}_q^{\vec{n},\vec{\alpha},\beta} \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)} \mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} \left(\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \right), \tag{2.3.9}$$

where

$$\mathcal{M}_{q,\vec{n}}^{\vec{\alpha}} = \prod_{i=1}^{r} \mathcal{M}_{q,n_i}^{\alpha_i}, \quad \mathcal{M}_{q,n_i}^{\alpha_i} = (\alpha_i)^{-s}\nabla^{n_i}(\alpha_i q^{n_i})^s, \tag{2.3.10}$$

and

$$\mathcal{G}_q^{\vec{n},\vec{\alpha},\beta} = (-1)^{|\vec{n}|}[-\beta]_q^{(|\vec{n}|)} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^{r} \frac{\alpha_i^{n_i} \prod_{j=1}^{n_i} q^{|\vec{n}|_i+\beta+j-1}}{\prod_{j=1}^{n_i} \left(\alpha_i q^{|\vec{n}|+\beta+j-1}-1\right)} \right) \left(\prod_{i=1}^{r} q^{n_i \sum_{j=i}^{r} n_j} \right),$$

where $|\vec{n}|_i = n_1 + \cdots + n_{i-1}$, $|\vec{n}|_1 = 0$.

Proof. We proceed in the same way as in Proposition (2.3.2).

For $i = 1, \ldots, r$, applying k_i-times the raising operators (2.3.8) in a recursive way one obtains

$$\prod_{i=1}^{r} \left(\frac{\alpha_i}{q^{k_i}}\right)^{-s} \nabla^{k_i} \alpha_i^s \frac{\Gamma_q(\beta+s)}{\Gamma_q(\beta)\Gamma_q(s+1)} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = [\beta-1]_q^{(|\vec{k}|)} q^{|\vec{k}|/2} \left(\prod_{i=1}^{r} \frac{\prod_{j=1}^{k_i} \left(\alpha_i q^{|\vec{n}|+\beta-j}-1\right)}{\alpha_i^{k_i}} \right) \times \prod_{i=1}^{r} q^{-n_i \sum_{j=i}^{r} k_j} \prod_{i=1}^{r-1} q^{-k_i \sum_{j=i+1}^{r} k_j} M_{q,\vec{n}+\vec{k}}^{\alpha_1/q^{k_1},\ldots,\alpha_r/q^{k_r},\beta-|\vec{k}|}(s) \frac{\Gamma_q(\beta-|\vec{k}|+s)}{\Gamma_q(\beta-|\vec{k}|)\Gamma_q(s+1)}.$$

Finally, taking $n_1 = n_2 = \cdots = n_r = 0$, $\beta = \beta + |\vec{k}|$, $\alpha_i = \alpha_i q^{k_i}$ and $k_i = n_i$, $i = 1, \ldots, r$, in the above expression, (2.3.9) holds. □

2.3.3 q-Meixner multiple orthogonal polynomials of the second kind

Definition 2.3.5. A polynomial $M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ that verifies the orthogonality conditions

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)[s]_q^{(k)} \upsilon_q^{\beta_i,\alpha}(s) \bigtriangleup x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r, \tag{2.3.11}$$

(see (1.3.1) with respect to the measures (2.2.4)) is said to be the q-Meixner multiple orthogonal polynomial of the second kind.

Again here the orthogonality conditions (1.3.1) have been written more conveniently as (2.3.11). Observe that, one can recover the multiple Meixner polynomials of the second kind given in [10] as a limiting case. In addition, it yields the property of the AT-system of positive discrete measures, resulting in the desired situation where the q-Meixner multiple orthogonal polynomial of the second kind $M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^+$ (see [10, Theorem 2.1, pp. 26–27]).

Replacing $[s]_q^{(k)}$ in (2.3.11) by (2.3.3) one has

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\nabla[s+1]_q^{(k+1)} \upsilon_q^{\beta_i,\alpha}(s) \bigtriangleup x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r.$$

Using (2.3.2), one gets

$$\begin{aligned}\sum_{s=0}^{\infty} \nabla \left[M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\upsilon_q^{\beta_i,\alpha}(s)\right] [s]_q^{(k+1)} \bigtriangleup x(s-1/2) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\upsilon_q^{\beta_i,\alpha}(s)\Delta[s]_q^{(k+1)} \bigtriangleup x(s-1/2) \\ &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\upsilon_q^{\beta_i,\alpha}(s)\nabla[s+1]_q^{(k+1)} \bigtriangleup x(s-1/2).\end{aligned}$$

Thus

$$\sum_{s=0}^{\infty} \nabla \left[M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\upsilon_q^{\beta_i,\alpha}(s)\right] [s]_q^{(k+1)} \bigtriangleup x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r.$$

On the other hand we have

$$\begin{aligned}\nabla \left[M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\upsilon_q^{\beta_i,\alpha}(s)\right] &= -\frac{\alpha^{-1}q^{-|\vec{n}|}q^{1/2}\left(1-\alpha q^{|\vec{n}|+\beta_i-1}\right)}{x(\beta_i-1)}\upsilon_q^{\beta_i-1,\alpha/q}(s)\left(x^{|\vec{n}|+1}+\cdots\right) \\ &= -\frac{\alpha^{-1}q^{-|\vec{n}|}q^{1/2}\left(1-\alpha q^{|\vec{n}|+\beta_i-1}\right)}{x(\beta_i-1)}\upsilon_q^{\beta_i-1,\alpha/q}(s)\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s).\end{aligned}$$

Consequently

$$\sum_{s=0}^{\infty} \mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s)\upsilon_q^{\beta_i-1,\alpha/q}(s)[s]_q^{(k+1)} \triangle x(s-1/2) = \sum_{s=0}^{\infty} \nabla \left[M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\upsilon_q^{\beta_i,\alpha}(s)\right] [s]_q^{(k+1)} \triangle x(s-1/2) = 0.$$

Then from (2.3.11) we deduce that $\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s) = M_{q,\vec{n}+\vec{e}_i}^{\vec{\beta}-\vec{e}_i,\alpha/q}(s)$. Therefore

$$\nabla \left[M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\upsilon_q^{\beta_i,\alpha}(s)\right] = -\frac{\alpha^{-1}q^{-|\vec{n}|}q^{1/2}\left(1-\alpha q^{|\vec{n}|+\beta_i-1}\right)}{x(\beta_i-1)}\upsilon_q^{\beta_i-1,\alpha/q}(s)M_{q,\vec{n}+\vec{e}_i}^{\vec{\beta}-\vec{e}_i,\alpha/q}(s).$$

For monic q-Meixner multiple orthogonal polynomials of the second kind we have r raising operators.

$$\mathcal{D}_q^{\beta_i,\alpha} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = -q^{1/2} M_{q,\vec{n}+\vec{e}_i}^{\vec{\beta}-\vec{e}_i,\alpha/q}(s), \tag{2.3.12}$$

where

$$\mathcal{D}_q^{\beta_i,\alpha} \stackrel{\text{def}}{=} \left(\frac{\alpha x\left(\beta_i-1\right)q^{|\vec{n}|}}{\left(1-\alpha q^{|\vec{n}|+\beta_i-1}\right)\upsilon_q^{\beta_i-1,\alpha/q}(s)} \nabla \upsilon_q^{\beta_i,\alpha}(s)\right).$$

In general,

$$\begin{aligned} q^{-|\vec{n}|-1/2}\mathcal{D}_q^{\beta_i,\alpha} f(s) = {} & \frac{1}{1-\alpha q^{|\vec{n}|+\beta_i-1}}\left[\alpha q^{\beta_i-1}\left(x(s)-x(1-\beta_i)\right)-x(s)\right] f(s) \\ & + \frac{1}{1-\alpha q^{|\vec{n}|+\beta_i-1}} x(s) \bigtriangledown f(s), \end{aligned} \tag{2.3.13}$$

holds for any function $f(s)$ defined on the discrete variable s. Notice that we call $\mathcal{D}_q^{\beta_i,\alpha}$ a raising operator since the i-th component of the multi-index $\vec{n}$ in (2.3.12) is increased by 1.

In the sequel we will only consider monic q-Meixner multiple orthogonal polynomials of the second kind.

Proposition 2.3.6. *There holds the following finite-difference analogue of the Rodrigues-type formula*

$$M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = \mathcal{G}_q^{\vec{n},\vec{\beta},\alpha}\frac{\Gamma_q(s+1)}{\alpha^s}\bar{\mathcal{M}}_{\vec{n}}^{\vec{\beta}}\left(\frac{\left(\alpha q^{|\vec{n}|}\right)^s}{\Gamma_q(s+1)}\right), \tag{2.3.14}$$

where

$$\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}} = \prod_{i=1}^{r}\bar{\mathcal{M}}_{q,n_i}^{\beta_i}, \quad \bar{\mathcal{M}}_{q,n_i}^{\beta_i} = \frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i+s)}\nabla^{n_i}\frac{\Gamma_q(\beta_i+n_i+s)}{\Gamma_q(\beta_i+n_i)}, \tag{2.3.15}$$

and

$$\mathcal{G}_q^{\vec{n},\vec{\beta},\alpha} = (-1)^{|\vec{n}|}\left(\alpha q^{|\vec{n}|}\right)^{|\vec{n}|} q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\frac{\prod\limits_{j=1}^{n_i} q^{\beta_i+j-1}}{\prod\limits_{j=1}^{n_i}\left(\alpha q^{|\vec{n}|+\beta_i+j-1}-1\right)}\right)\left(\prod_{i=1}^{r}[-\beta_i]_q^{(n_i)}\right).$$

Proof. The proof follows the same patterns given in Proposition (2.3.2) adapted to the new operator $\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}$.

For $i = 1, \ldots, r$ applying k_i-times the raising operators (2.3.12) in a recursive way one obtains

$$\prod_{i=1}^{r} \frac{\Gamma(\beta_i - k_i)}{\Gamma(\beta_i - k_i + s)} \nabla^{k_i} \frac{\Gamma_q(\beta_i + s)}{\Gamma_q(\beta_i)} \frac{(\alpha)^s}{\Gamma_q(s+1)} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = \prod_{i=1}^{r} [\beta_i - 1]_q^{(k_i)} q^{|\vec{k}|/2} q^{-(|\vec{k}|)|\vec{n}|}$$
$$\times \left(\prod_{i=1}^{r} \frac{\prod_{j=1}^{k_i} \left(\alpha q^{|\vec{n}|+\beta_i - j} - 1\right)}{\alpha^{k_i}} \right) M_{q,\vec{n}+\vec{k}}^{\beta_1 - k_1, \ldots, \beta_r - k_r, \alpha/q^{|\vec{k}|}}(s) \frac{(\alpha/q^{|\vec{k}|})^s}{\Gamma_q(s+1)}.$$

Finally, taking $n_1 = n_2 = \cdots = n_r = 0$, $\beta_i = \beta_i + k_i$, $\alpha = \alpha q^{|\vec{k}|}$ and $k_i = n_i$, $i = 1, \ldots, r$, in the above expression, (2.3.14) holds. □

2.3.4 q-Kravchuk multiple orthogonal polynomials

Definition 2.3.7. A polynomial $K_{q,\vec{n}}^{\vec{p},N}(s)$, with multi-index $\vec{n} \in \mathbb{N}^r$ and degree $|\vec{n}|$ that verifies the orthogonality conditions

$$\sum_{s=0}^{N} K_{q,\vec{n}}^{\vec{p},N}(s) [s]_q^{(k)} \upsilon_q^{p_i,N}(s) \triangle x(s - 1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r, \tag{2.3.16}$$

(see (1.3.1) with respect to the measures (2.2.12)) is said to be the q-Kravchuk multiple orthogonal polynomial.

The orthogonality conditions (1.3.1) have been rewritten by using the canonical basis (2.3.3) as (2.3.16). Here, one can recover the multiple Kravchuk polynomials given in [10] as a limiting case when q approaches 1. Observe that, we have an AT-system of positive discrete measures, consequently the q-Kravchuk multiple orthogonal polynomial $K_{q,\vec{n}}^{\vec{p},N}(s)$ has exactly $|\vec{n}|$ different zeros on $\mathbb{R}^+$ (see [10, Theorem 2.1, pp. 26–27]).

In this case, replacing $[s]_q^{(k)}$ in (2.3.16) by (2.3.3) one has

$$\sum_{s=0}^{N} K_{q,\vec{n}}^{\vec{p},N}(s) \nabla [s+1]_q^{(k+1)} \upsilon_q^{p_i,N}(s) \triangle x(s - 1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r.$$

Using (2.3.2), one gets

$$\sum_{s=0}^{N} \nabla \left[K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \right] [s]_q^{(k+1)} \triangle x(s-1/2) = - \sum_{s=0}^{N} K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \Delta [s]_q^{(k+1)} \triangle x(s - 1/2)$$
$$= - \sum_{s=0}^{N} K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \nabla [s+1]_q^{(k+1)} \triangle x(s - 1/2).$$

Thus

$$\sum_{s=0}^{N} \nabla \left[K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \right] [s]_q^{(k+1)} \triangle x(s-1/2) = 0, \qquad 0 \le k \le n_i - 1, \qquad i = 1, \ldots, r.$$

On the other hand we have

$$\nabla \left[K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \right] = q^{-s+1/2} \left[\upsilon_q^{p_i,N}(s) K_{q,\vec{n}}^{\vec{p},N}(s) - \upsilon_q^{p_i,N}(s-1) K_{q,\vec{n}}^{\vec{p},N}(s-1) \right],$$

where

$$\upsilon_q^{p_i,N}(s) = q^{-s} \frac{[x(N+1) - x(s)]}{(1-p_i)\,[N+1]_q} \upsilon_q^{p_i,N+1}(s)$$

and

$$\upsilon_q^{p_i,N}(s-1) = q^{-s} \frac{q x(s)}{p_i\,[N+1]_q} \upsilon_q^{p_i,N+1}(s).$$

Observe that the coefficient $q^{\binom{s}{2}}$ in $\upsilon_q^{p_i,N}(s)$ is necessary to obtain the coefficient q^{-s} in the above expression and thus to obtain the polynomial $x^{|\vec{n}|+1} + \textit{lower terms}$ in the following expression

$$\begin{aligned}
\nabla \left[K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \right] &= -\frac{p_i^{-1} \beta_i^{-1} q^{-(|\vec{n}|+2N+1)} q^{1/2} \left[p_i \left(q^{|\vec{n}|-1} - 1 \right) + 1 \right]}{[N+1]_q} \upsilon_q^{p_i/q,N+1,q^2\beta_i}(s) \left(x^{|\vec{n}|+1} + \cdots \right) \\
&= -\frac{p_i^{-1} \beta_i^{-1} q^{-(|\vec{n}|+2N+1)} q^{1/2} \left[p_i \left(q^{|\vec{n}|-1} - 1 \right) + 1 \right]}{[N+1]_q} \upsilon_q^{p_i/q,N+1,q^2\beta_i}(s) \mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s),
\end{aligned}$$

where $\beta_i = 1 - p_i$.
Consequently

$$\begin{aligned}
&\sum_{s=0}^{N} \mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s) \upsilon_q^{\alpha_i/q,\beta-1}(s) [s]_q^{(k+1)} \triangle x(s-1/2) \\
&\qquad = \sum_{s=0}^{N} \nabla \left[K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \right] [s]_q^{(k+1)} \triangle x(s-1/2) = 0.
\end{aligned}$$

Then from (2.3.16) we deduce that $\mathcal{Q}_{q,\vec{n}+\vec{e}_i}(s) = K_{q,\vec{n}+\vec{e}_i}^{\vec{p},N+1,q^2\beta_i}(s)$. Therefore

$$\nabla \left[K_{q,\vec{n}}^{\vec{p},N}(s) \upsilon_q^{p_i,N}(s) \right] = -\frac{p_i^{-1} \beta_i^{-1} q^{-(|\vec{n}|+2N+1)} q^{1/2} \left[p_i \left(q^{|\vec{n}|-1} - 1 \right) + 1 \right]}{[N+1]_q} \upsilon_q^{p_i/q,N+1,q^2\beta_i}(s) K_{q,\vec{n}+\vec{e}_i}^{\vec{p},N+1,q^2\beta_i}(s).$$

For monic q-Kravchuk multiple orthogonal polynomials we have r raising operators

$$\mathcal{D}_q^{p_i,N} K_{q,\vec{n}}^{\vec{p},N}(s) = -q^{1/2} K_{q,\vec{n}+\vec{e}_i}^{\vec{p},N+1,q^2\beta_i}(s), \qquad i = 1, \ldots, r, \tag{2.3.17}$$

where

$$\mathcal{D}_q^{p_i,N} \overset{\text{def}}{=} \left(\frac{p_i \beta_i q^{|\vec{n}|+2N+1} [N+1]_q}{\left[p_i \left(q^{|\vec{n}|-1} - 1\right) + 1\right] \upsilon_q^{p_i,N+1,q^2\beta_i}(s)} \nabla \upsilon_q^{p_i,N}(s) \right).$$

Furthermore,

$$q^{-|\vec{n}|-1/2} \mathcal{D}_q^{p_i,N} f(s) = \frac{q^{-1}}{\left[p_i \left(q^{|\vec{n}|-1} - 1\right) + 1\right]} \left[p_i \left(x(N+1) - x(s)\right) - (1-p_i) q x(s)\right] f(s) + \frac{(1-p_i)}{\left[p_i \left(q^{|\vec{n}|-1} - 1\right) + 1\right]} x(s) \bigtriangledown f(s),$$

for any function $f(s)$ defined on the discrete variable s. Notice that we call $\mathcal{D}_q^{p_i,N}$ a raising operator since the i-th component of the multi-index $\vec{n}$ in (2.3.17) is increased by 1.

In the sequel we will only consider monic q-Kravchuk multiple orthogonal polynomials.

Proposition 2.3.8. *There holds the following q-analogue of Rodrigues-type formula*

$$K_{q,\vec{n}}^{\vec{p},N}(s) = \mathcal{G}_q^{\vec{n},\vec{p},N} \frac{\Gamma_q(N-s+1)\Gamma_q(s+1)}{q^{\binom{s}{2}} [N]_q!} \mathcal{K}_{q,\vec{n}}^{\vec{p}} \left(\frac{q^{\binom{s}{2}} [N-|\vec{n}|]_q!}{\Gamma_q(N-|\vec{n}|-s+1)\Gamma_q(s+1)} \right), \tag{2.3.18}$$

where

$$\mathcal{K}_{q,\vec{n}}^{\vec{p}} = \prod_{i=1}^{r} \mathcal{K}_{q,n_i}^{p_i}, \quad \mathcal{K}_{q,n_i}^{p_i} = \left(\frac{p_i}{1-p_i}\right)^{-s} \nabla^{n_i} \left(\frac{p_i q^{2n_i}}{1-p_i}\right)^{s}, \tag{2.3.19}$$

and

$$\mathcal{G}_q^{\vec{n},\vec{p},N} = (-1)^{|\vec{n}|} [N]_q^{(|\vec{n}|)} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^{r} \frac{p_i^{n_i}}{\prod\limits_{j=1}^{n_i} q^{-j} \left[p_i \left(q^{|\vec{n}|-|\vec{n}|_i-j-1} - 1\right) + 1\right]} \right) q^{-2|\vec{n}|} \left(\prod_{i=1}^{r-1} q^{n_i \sum\limits_{j=i+1}^{r} n_j} \right),$$

where $|\vec{n}|_i = n_1 + \cdots + n_{i-1}$, $|\vec{n}|_1 = 0$.

Proof. Highlight the steps following in the proof of Proposition (2.3.2).

For $i = 1, \ldots, r$, applying k_i-times the raising operators (2.3.17) in a recursive way one obtains

$$\prod_{i=1}^{r} \left(\frac{p_i q^{-2k_i}}{1-p_i}\right)^{-s} \nabla^{n_i} \left(\frac{p_i}{1-p_i}\right)^{s} \frac{q^{\binom{s}{2}} [N]_q!}{\Gamma_q(N-s+1)\Gamma_q(s+1)} K_{q,\vec{n}}^{\vec{p},N}(s) = [N+1]_q^{(|\vec{k}|)} q^{|\vec{k}|/2} \left(\prod_{i=1}^{r} \frac{\prod\limits_{j=1}^{k_i} \left(\alpha_i q^{|\vec{n}|+\beta-j} - 1\right)}{p_i^{k_i} (1-p_i)^{k_i}} \right) \prod_{i=1}^{r} q^{-n_i \sum\limits_{j=i}^{r} k_j} \prod_{i=1}^{r-1} q^{-k_i \sum\limits_{j=i+1}^{r} k_j} K_{q,\vec{n}+\vec{k}}^{p_1,\ldots,p_r,N+|\vec{k}|,q^{2k_1}\beta_1,\ldots,q^{2k_r}\beta_r}(s) \frac{q^{\binom{s}{2}} [N+|\vec{k}|]_q!}{\Gamma_q(N+|\vec{k}|-s+1)\Gamma_q(s+1)}.$$

Finally, taking $n_1 = n_2 = \cdots = n_r = 0$, $N = N - |\vec{k}|$, $\beta_i = \beta_i q^{-2k_i}$ and $k_i = n_i$, $i = 1, \ldots, r$, in the above expression, (2.3.18) holds.

$\square$

Chapter 3

Algebraic properties

In this chapter we study two important results obtained in this thesis, namely some properties of q-multiple orthogonal polynomials, the q-Difference equation and q-Recurrence relation.

3.1 q-Difference equation

The strategy that we will follow to deduce the $(r+1)$-order q-difference equation is the following.

First step. Define an r-dimensional subspace $\mathbb{V}$ of polynomials on the variable $x(s)$ of degree at most $|\vec{n}|-1$ by means of interpolatory conditions.

Second step. Find the lowering operator and express its action on the polynomials as a linear combination of the basis elements of $\mathbb{V}$. Thus, it depends on the treated space of family.

Third step. Combine the lowering and the raising operators. In particular we will use the lowering operators (2.3.4), (2.3.8), (2.3.12), and (2.3.17), respectively to get an $(r+1)$-order q-difference equation after a suitable combination with lowering operators (in the same fashion that [9, 11], [13], and [65]). Other approaches have been considered in [27, 93].

These steps represent the general features of the algebraic approach we will use in this section. However, some 'ad hoc' computations are needed because of the dependence of the explicit expressions involved in the above steps on the given family of multiple orthogonal polynomials.

3.1.1 q-Charlier multiple orthogonal polynomials

In this section we will find a lowering operator for the q-Charlier multiple orthogonal polynomials. Then, we will combine it with the raising operators (2.3.4) to get an $(r+1)$-order q-difference equation.

Lemma 3.1.1. *Let $\mathbb{V}$ be the linear subspace of polynomials $Q(s)$ on the lattice $x(s)$ of degree at most $|\vec{n}|-1$ defined by the following conditions*

$$\sum_{s=0}^{\infty} Q(s)[s]_q^{(k)} v_q^{q\alpha_j}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \leq k \leq n_j - 2 \qquad \textit{and} \qquad j = 1, \ldots, r.$$

Then, the system $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$, where $\vec{\alpha}_{i,q} = (\alpha_1, \ldots, q\alpha_i, \ldots, \alpha_r)$, is a basis for $\mathbb{V}$.

Proof. From orthogonality relations

$$\sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(k)}\upsilon_q^{q\alpha_j}(s)\bigtriangledown x_1(s)=0, \qquad 0\le k\le n_j-2, \qquad j=1,\dots,r,$$

we have that polynomials $C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)$, $i=1,\dots,r$, belong to $\mathbb{V}$.

Now, aimed to get a contradiction, let us assume that there exist constants λ_i, $i=1,\dots,r$, such that

$$\sum_{i=1}^{r}\lambda_i C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)=0, \qquad \text{where} \qquad \sum_{i=1}^{r}|\lambda_i|>0.$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)}\upsilon_q^{\alpha_k}(s)\bigtriangledown x_1(s)$ and then taking summation on s from 0 to ∞, one gets

$$\sum_{i=1}^{r}\lambda_i\sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)}\upsilon_q^{\alpha_k}(s)\bigtriangledown x_1(s)=0.$$

Thus, from the relations

$$\sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)}\upsilon_q^{\alpha_k}(s)\bigtriangledown x_1(s)=c\delta_{i,k}, \qquad c\in\mathbb{R}\setminus\{0\}, \tag{3.1.1}$$

we deduce that $\lambda_k=0$ for $k=1,\dots,r$. Here $\delta_{i,k}$ represents the Kronecker delta symbol. Therefore, $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$. Furthermore, we know that any polynomial of $\mathbb{V}$ can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}|-r)$ linear conditions are imposed on $\mathbb{V}$. Consequently the dimension of $\mathbb{V}$ is at most r. Hence, the system $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$ spans $\mathbb{V}$, which completes the proof. □

Now we will prove that the operator (1.3.4) is indeed a lowering operator for the sequence of q-Charlier multiple orthogonal polynomials $C_{q,\vec{n}}^{\vec{\alpha}}(s)$.

Lemma 3.1.2. *There holds the following relation*

$$\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)=\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2}[n_i]_q^{(1)} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s). \tag{3.1.2}$$

Proof. Using summation by parts we have

$$\begin{aligned}\sum_{s=0}^{\infty}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(k)}\upsilon_q^{q\alpha_j}(s)\bigtriangledown x_1(s) &= -\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)\nabla\left[[s]_q^{(k)}\upsilon_q^{q\alpha_j}(s)\right]\bigtriangledown x_1(s)\\ &= -\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)\varphi_{j,k}(s)\upsilon_q^{\alpha_j}(s)\bigtriangledown x_1(s),\end{aligned} \tag{3.1.3}$$

where

$$\varphi_{j,k}(s) = q^{1/2}[s]_q^{(k)} - q^{-1/2}\frac{x(s)}{\alpha_j}[s-1]_q^{(k)},$$

is a polynomial of degree $\leq k+1$ in the variable $x(s)$. Consequently, from the orthogonality conditions (2.3.1) we get

$$\sum_{s=0}^{\infty} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(k)} \upsilon_q^{q\alpha_j}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \leq k \leq n_j - 2, \qquad j = 1, \ldots, r.$$

Hence, from Lemma 3.1.1, $\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s) \in \mathbb{V}$. Moreover, $\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)$ can univocally be expressed as a linear combination of polynomials $\{C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)\}_{i=1}^{r}$, i.e.

$$\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s) = \sum_{i=1}^{r} \xi_i C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s), \qquad \sum_{i=1}^{r} |\xi_i| > 0. \tag{3.1.4}$$

Multiplying both sides of the equation (3.1.4) by $[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s)$ and using relations (3.1.1) one has

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s) &= \sum_{i=1}^{r} \xi_i \sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s) \\ &= \xi_k \sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s). \end{aligned} \tag{3.1.5}$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of equation (3.1.3), then left-hand side of equation (3.1.5) transforms into

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s) &= -\sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s) \varphi_{k,n_k-1}(s) \upsilon_q^{\alpha_k}(s) \bigtriangledown x_1(s) \\ &= \frac{q^{-1/2}}{\alpha_k} \sum_{s=0}^{\infty} C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k)} \upsilon_q^{\alpha_k}(s) \bigtriangledown x_1(s). \end{aligned} \tag{3.1.6}$$

Here we have used that $x(s)[s-1]_q^{(n_k-1)} = [s]_q^{(n_k)}$ to get

$$\varphi_{k,n_k-1}(s) = -(q^{-1/2}/\alpha_k)[s]_q^{(n_k)} + \text{lower degree terms.}$$

On the other hand, from (2.3.4) one has that

$$\frac{1}{\alpha_k} \upsilon_q^{\alpha_k}(s) C_{q,\vec{n}}^{\vec{\alpha}}(s) = -q^{|\vec{n}|-1/2} \nabla \left[\upsilon_q^{q\alpha_k}(s) C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q}}(s) \right]. \tag{3.1.7}$$

Then, by conveniently substituting (3.1.7) in the right-hand side of equation (3.1.6) and using once more summation by parts, we get

$$\sum_{s=0}^{\infty} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s) = -q^{|\vec{n}|-1} \sum_{s=0}^{\infty} [s]_q^{(n_k)} \nabla \left[\upsilon_q^{q\alpha_k}(s) C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q}}(s) \right] \bigtriangledown x_1(s)$$
$$= q^{|\vec{n}|-1} \sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q}}(s) \Delta \left[[s]_q^{(n_k)} \right] \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s).$$

Since $\Delta[s]_q^{(n_k)} = q^{3/2-n_k}[n_k]_q^{(1)}[s]_q^{(n_k-1)}$ we finally have

$$\sum_{s=0}^{\infty} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s) = q^{|\vec{n}|-n_k+1/2}[n_k]_q^{(1)} \sum_{s=0}^{\infty} C_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q}}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k}(s) \bigtriangledown x_1(s),$$

where

$$[n_k]_q^{(1)} = x(n_k) = \frac{q^{n_k}-1}{q-1}. \tag{3.1.8}$$

Therefore, comparing this equation with (3.1.5) we obtain the coefficients in the expansion (3.1.4), i.e.

$$\xi_k = q^{|\vec{n}|-n_k+1/2}[n_k]_q^{(1)},$$

which proves relation (3.1.2). □

Theorem 3.1.3. *The q-Charlier multiple orthogonal polynomial $C_{q,\vec{n}}^{\vec{\alpha}}(s)$ satisfies the following $(r+1)$-order q-difference equation*

$$\prod_{i=1}^{r} \mathcal{D}_q^{q\alpha_i} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s) = -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1}[n_i]_q^{(1)} \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{q\alpha_j} C_{q,\vec{n}}^{\vec{\alpha}}(s). \tag{3.1.9}$$

Proof. Since operators (2.3.4) commute, we write

$$\prod_{i=1}^{r} \mathcal{D}_q^{q\alpha_i} = \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{q\alpha_j} \right) \mathcal{D}_q^{q\alpha_i}, \tag{3.1.10}$$

and then using (2.3.4), by acting on equation (3.1.2) with the product of operators (3.1.10), we obtain the following relation

$$\prod_{i=1}^{r} \mathcal{D}_q^{q\alpha_i} \Delta C_{q,\vec{n}}^{\vec{\alpha}}(s) = \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2}[n_i]_q^{(1)} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{q\alpha_j} \right) \mathcal{D}_q^{q\alpha_i} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q}}(s)$$
$$= -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1}[n_i]_q^{(1)} \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{q\alpha_j} C_{q,\vec{n}}^{\vec{\alpha}}(s),$$

which proves (3.1.9). □

3.1.2 q-Meixner multiple orthogonal polynomials of the first kind

In this section we will find a lowering operator for the q-Meixner multiple orthogonal polynomials of the first kind. Again here we will follow the same strategy already developed in Section (3.1.1).

Lemma 3.1.4. *Let $\mathbb{V}$ be the linear subspace of polynomials $Q(s)$ on the lattice $x(s)$ of degree at most $|\vec{n}|-1$ defined by the following conditions*

$$\sum_{s=0}^{\infty} Q(s)[s]_q^{(k)} \upsilon_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2 \qquad \textit{and} \qquad j = 1,\dots,r.$$

Then, the system $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\}_{i=1}^{r}$, where $\vec{\alpha}_{i,q} = (\alpha_1,\dots,q\alpha_i,\dots,\alpha_r)$, is a basis for $\mathbb{V}$.

Proof. From orthogonality relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)[s]_q^{(k)} \upsilon_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2, \qquad j = 1,\dots,r,$$

we have that polynomials $M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)$, $i = 1,\dots,r$, belong to $\mathbb{V}$.

Now, aimed to get a contradiction, let us assume that there exist constants λ_i, $i = 1,\dots,r$, such that

$$\sum_{i=1}^{r} \lambda_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s) = 0, \qquad \text{where} \qquad \sum_{i=1}^{r} |\lambda_i| > 0.$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)} \upsilon_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s)$ and then taking summation on s from 0 to ∞, one gets

$$\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)[s]_q^{(n_k-1)} \upsilon_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s) = 0.$$

Thus, from relations

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)[s]_q^{(n_k-1)} \upsilon_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s) = c\delta_{i,k}, \qquad c \in \mathbb{R} \setminus \{0\}, \tag{3.1.11}$$

we deduce that $\lambda_k = 0$ for $k = 1,\dots,r$. Here $\delta_{i,k}$ represents the Kronecker delta symbol. Therefore, $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$. Furthermore, we know that any polynomial of $\mathbb{V}$ can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}|-r)$ linear conditions are imposed on $\mathbb{V}$. Consequently the dimension of $\mathbb{V}$ is at most r. Hence, the system $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\}_{i=1}^{r}$ spans $\mathbb{V}$, which completes the proof. $\square$

Now we will prove that the operator (1.3.4) is indeed a lowering operator for the sequence of q-Meixner multiple orthogonal polynomials of the first kind $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$.

Lemma 3.1.5. *There holds the following relation*

$$\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2} \frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}} [n_i]_q^{(1)} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s). \tag{3.1.12}$$

Proof. Using summation by parts we have

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(k)} \upsilon_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \nabla \left[[s]_q^{(k)} \upsilon_q^{q\alpha_j,\beta+1}(s)\right] \bigtriangledown x_1(s) \\ &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \varphi_{j,k}(s) \upsilon_q^{\alpha_j,\beta}(s) \bigtriangledown x_1(s), \end{aligned} \tag{3.1.13}$$

where

$$\varphi_{j,k}(s) = q^{1/2} \left(\frac{q^{\beta} x(s)}{x(\beta)} + 1 \right) [s]_q^{(k)} - q^{-1/2} \frac{x(s)}{\alpha_j x(\beta)} [s-1]_q^{(k)},$$

is a polynomial of degree $\leq k+1$ in the variable $x(s)$. Consequently, from the orthogonality conditions (2.3.7) we get

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(k)} \upsilon_q^{q\alpha_j,\beta+1}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \leq k \leq n_j - 2, \qquad j = 1, \ldots, r.$$

Hence, from Lemma 3.1.4, $\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \in \mathbb{V}$. Moreover, $\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ can univocally be expressed as a linear combination of polynomials $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\}_{i=1}^{r}$, i.e.

$$\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \sum_{i=1}^{r} \xi_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s), \qquad \sum_{i=1}^{r} |\xi_i| > 0. \tag{3.1.14}$$

Multiplying both sides of the equation (3.1.14) by $[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s)$ and using relations (3.1.11) one has

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s) &= \sum_{i=1}^{r} \xi_i \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s) \\ &= \xi_k \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s). \end{aligned} \tag{3.1.15}$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of equation (3.1.13), then left-hand side of equation (3.1.15) transforms into

$$\begin{aligned}\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)} \upsilon_q^{q\alpha_k,\beta+1}(s) \bigtriangledown x_1(s) &= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) \varphi_{k,n_k-1}(s) \upsilon_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s) \\ &= \frac{q^{-1/2}\left(1-\alpha_k q^{n_k+\beta}\right)}{\alpha_k x(\beta)} \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k)} \upsilon_q^{\alpha_k,\beta}(s) \bigtriangledown x_1(s). \end{aligned} \tag{3.1.16}$$

Here we have used that $x(s)[s-1]_q^{(n_k-1)}=[s]_q^{(n_k)}$ to get

$$\varphi_{k,n_k-1}(s)=-\frac{q^{-1/2}\left(1-\alpha_k q^{n_k+\beta}\right)}{\alpha_k x(\beta)}[s]_q^{(n_k)}+\text{lower degree terms.}$$

On the other hand, from (2.3.8) one has that

$$\frac{q^{-1/2}\left(1-\alpha_k q^{|\vec{n}|+\beta}\right)}{\alpha_k x(\beta)}\upsilon_q^{\alpha_k,\beta}(s)M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)=-q^{|\vec{n}|-1/2}\nabla\left[\upsilon_q^{q\alpha_k,\beta+1}(s)M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)\right]. \quad (3.1.17)$$

Then, by conveniently substituting (3.1.17) in the right-hand side of equation (3.1.16) and using once more summation by parts, we get

$$\begin{aligned}\sum_{s=0}^{\infty}\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k,\beta+1}(s)\bigtriangledown x_1(s)&\\ =-q^{|\vec{n}|-1}\frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}}\sum_{s=0}^{\infty}[s]_q^{(n_k)}\nabla\left[\upsilon_q^{q\alpha_k,\beta+1}(s)M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)\right]\bigtriangledown x_1(s)&\\ =q^{|\vec{n}|-1}\frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}}\sum_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)\Delta\left[[s]_q^{(n_k)}\right]\upsilon_q^{q\alpha_k,\beta+1}(s)\bigtriangledown x_1(s).&\end{aligned}$$

Since $\Delta[s]_q^{(n_k)}=q^{3/2-n_k}[n_k]_q^{(1)}[s]_q^{(n_k-1)}$ we finally have

$$\begin{aligned}\sum_{s=0}^{\infty}\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k,\beta+1}(s)\bigtriangledown x_1(s)&\\ =q^{|\vec{n}|-n_k+1/2}\frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}}[n_k]_q^{(1)}\sum_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_k}^{\vec{\alpha}_{k,q},\beta+1}(s)[s]_q^{(n_k-1)}\upsilon_q^{q\alpha_k,\beta+1}(s)\bigtriangledown x_1(s).&\end{aligned}$$

Therefore, comparing this equation with (3.1.15) we obtain the coefficients in the expansion (3.1.14), i.e.

$$\xi_k=q^{|\vec{n}|-n_k+1/2}\frac{1-\alpha_k q^{n_k+\beta}}{1-\alpha_k q^{|\vec{n}|+\beta}}[n_k]_q^{(1)},$$

which proves relation (3.1.12). □

Theorem 3.1.6. *The q-Meixner multiple orthogonal polynomial of the first kind $M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ satisfies the following $(r+1)$-order q-difference equation*

$$\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1+i-r}\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)=-\sum_{i=1}^{r}q^{|\vec{n}|-n_i+1}\frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}}[n_i]_q^{(1)}\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1+j-r}M_{q,\vec{n}}^{\vec{\alpha},\beta}(s). \quad (3.1.18)$$

Proof. Since the operators (2.3.8) commute, we write

$$\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1+i-r} = \left(\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1+j-r}\right)\mathcal{D}_q^{q\alpha_i,\beta+1}, \tag{3.1.19}$$

and then using (2.3.8), by acting on equation (3.1.12) with the product of operators (3.1.19), we obtain the following relation

$$\begin{aligned}
&\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1+i-r}\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)\\
&\qquad= \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2}\frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}}[n_i]_q^{(1)}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1+j-r}\right)\mathcal{D}_q^{q\alpha_i,\beta+1}M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}_{i,q},\beta+1}(s)\\
&\qquad\qquad= -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1}\frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}}[n_i]_q^{(1)}\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1+j-r}M_{q,\vec{n}}^{\vec{\alpha},\beta}(s),
\end{aligned}$$

which proves (3.1.18). □

3.1.3 q-Meixner multiple orthogonal polynomials of the second kind

In this section we will find a lowering operator for the q-Meixner multiple orthogonal polynomials of the second kind. Again here we will follow the same strategy already developed in Section (3.1.1).

Lemma 3.1.7. *The q-Meixner multiple orthogonal polynomials of the second kind satisfy the following property*

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\,[s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,q\alpha}(s)\bigtriangledown x_1(s) = \tilde{a}_{k,i}\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}}^{\vec{\beta}+\vec{e},q^r\alpha}(s)\,[s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,q^r\alpha}(s)\bigtriangledown x_1(s),$$

where

$$\tilde{a}_{k,i} = \frac{1-\alpha q^{|\vec{n}|+\beta_i}}{\alpha q^{|\vec{n}|+\beta_i}}\frac{1}{x(n_k+\beta_k-\beta_i)}\prod_{j=1}^{r}\frac{\alpha q^{|\vec{n}|+\beta_j}}{1-\alpha q^{|\vec{n}|+\beta_j}}x(n_k+\beta_k-\beta_j),\quad k,i=1,2,\ldots,r, \tag{3.1.20}$$

and $\vec{e}=\sum_{i=1}^{r}\vec{e}_i$.

Proof. By shifting conveniently the parameters involved in (2.3.12) and (2.3.13), respectively, one has

$$\begin{aligned}
M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) &= -q^{-1/2}\mathcal{D}_q^{\beta_i+1,q\alpha}\left[M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\right]\\
&= -\frac{q^{|\vec{n}|-1}}{1-\alpha q^{|\vec{n}|+\beta_i}}\left\{\left[\alpha q^{\beta_i+1}\left(x(s)-x(-\beta_i)\right)-x(s)\right]M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)+x(s)\bigtriangledown M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\right\}.
\end{aligned}$$

Thus, we obtain

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\,[s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,\alpha}(s) \bigtriangledown x_1(s) = -\frac{q^{|\vec{n}|-1}}{1-\alpha q^{|\vec{n}|+\beta_i}} \sum_{s=0}^{\infty} [s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,\alpha}(s) \bigtriangledown x_1(s)$$
$$\times \left\{ \left[\alpha q^{\beta_i+1}\left(x(s)-x(-\beta_i)\right)-x(s)\right] M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s) + x(s) \bigtriangledown M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s) \right\}.$$

By using summation by parts in the above expression one gets

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\,[s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,\alpha}(s) \bigtriangledown x_1(s)$$
$$= \frac{\alpha q^{|\vec{n}|+\beta_i}}{1-\alpha q^{|\vec{n}|+\beta_i}} x(n_k+\beta_k-\beta_i) \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\,[s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s)$$
$$+ \frac{\alpha q^{|\vec{n}|-n_k+2}}{1-\alpha q^{|\vec{n}|+\beta_i}} x(\beta_k-1)x(n_k+\beta_k-1) \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\,[s]_q^{(n_k-2)}\,\upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s).$$

By orthogonality, we have

$$\sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\,[s]_q^{(n_k-2)}\,\upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s) = 0.$$

Thus, one gets

$$\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\,[s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,\alpha}(s) \bigtriangledown x_1(s) = \frac{\alpha q^{|\vec{n}|+\beta_i}}{1-\alpha q^{|\vec{n}|+\beta_i}} x(n_k+\beta_k-\beta_i)$$
$$\times \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\,[s]_q^{(n_k-1)}\,\upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s). \quad (3.1.21)$$

Then, by using inductively (3.1.21) the statement holds.

□

Lemma 3.1.8. *Let A be the following r-dimensional matrix*

$$A = \begin{pmatrix} \frac{1}{x(n_1)} & \frac{1}{x(n_1+\beta_1-\beta_2)} & \cdots & \frac{1}{x(n_1+\beta_1-\beta_r)} \\ \frac{1}{x(n_2+\beta_2-\beta_1)} & \frac{1}{x(n_2)} & \cdots & \frac{1}{x(n_2+\beta_2-\beta_r)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x(n_r+\beta_r-\beta_1)} & \frac{1}{x(n_r+\beta_r-\beta_2)} & \cdots & \frac{1}{x(n_r)} \end{pmatrix}$$

$$= \left(a_{i,j}\right)_{i,j=1}^{r}, \quad a_{i,j} = \frac{1}{x(n_i+\beta_i-\beta_j)},$$

then the determinant of A is

$$\det A = \frac{\prod_{k=1}^{r-1}\prod_{l=k+1}^{r} x(\beta_l-\beta_k)q^{n_l}x(n_k-n_l+\beta_k-\beta_l)}{\prod_{k=1}^{r}\prod_{l=1}^{r} x(n_l+\beta_l-\beta_k)}. \tag{3.1.22}$$

Here, to prove (3.1.22) *we will follow the operations indicated in [11, Lemma 3.2, p. 7].*

Proof. Let us proceed by column and row operations on the matrix A. Observe that, $k=1,\ldots,r$ and $i=1,\ldots,r$, the following relation

$$a_{k,i}-a_{k,1}=\lambda_{i,1}a_{k,i}a_{k,1}q^{n_k+\beta_k}, \quad \lambda_{i,1}=x(\beta_i-\beta_1)q^{-\beta_i}, \tag{3.1.23}$$

holds.

Therefore, based on (3.1.23) if A_k denotes the kth column of A, $k=1,\ldots,r$, one gets

$$\begin{aligned}\det A &= \det(A_1, A_2-A_1,\ldots,A_r-A_1)\\ &= \left(\prod_{k=1}^{r} a_{k,1}\right)\left(\prod_{i=2}^{r}\lambda_{i,1}\right)\begin{pmatrix}1 & c_{1,2} & \cdots & c_{1,r}\\ 1 & c_{2,2} & \cdots & c_{2,r}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & c_{r,2} & \cdots & c_{r,r}\end{pmatrix},\end{aligned}$$

where $c_{k,i}=a_{k,i}q^{n_k+\beta_k}$. Now, if one subtracts the first row from the other ones, and takes into account that

$$c_{k,i}-c_{k,1}=a_{k,i}a_{1,i}\mu_{k,1}, \quad \mu_{k,1}=x(n_1-n_k+\beta_1-\beta_k)q^{n_k+\beta_k}, \quad i,k=2,\ldots,r,$$

then

$$\begin{aligned}\det A &= \det(A_1, A_2-A_1,\ldots,A_r-A_1)\\ &= \left(\prod_{k=1}^{r} a_{k,1}\right)\left(\prod_{j=1}^{r} a_{1,j}\right)\left(\prod_{i=2}^{r}\lambda_{i,1}\mu_{i,1}\right)\begin{pmatrix}a_{2,2} & a_{2,3} & \cdots & a_{2,r}\\ a_{3,2} & a_{3,3} & \cdots & a_{3,r}\\ \vdots & \vdots & \ddots & \vdots\\ a_{r,2} & a_{r,3} & \cdots & a_{r,r}\end{pmatrix}.\end{aligned}$$

Finally, repeating the previous column and row operations but on lower dimensional matrices the expression (3.1.22) can be proved by induction. $\square$

Lemma 3.1.9. *Let $\mathbb{V}$ be the space of polynomials ϑ such that $\deg(\vartheta)\leq|\vec{n}|-1$ and*

$$\sum_{s\geq 0}\vartheta(s)\,[s]_q^{(k)}\,\upsilon_q^{\beta_j+1,q\alpha}(s)\bigtriangledown x_1(s)=0, \quad 0\leq k\leq n_j-2 \quad \textit{and} \quad j=1,2,\ldots,r.$$

Then, for the q-Meixner multiple orthogonal polynomials of the second kind $\left\{M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\right\}_{|\vec{n}|\geq 0}$, we have that the system $\left\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\right\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$.

Proof. From orthogonality relations

$$\sum_{s\geq 0} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\,[s]_q^{(k)}\,\upsilon_q^{\beta_j+1,q\alpha}(s) \bigtriangledown x_1(s)=0, \quad 0\leq k\leq n_j-2, \quad j=1,2,\ldots,r,$$

we have that polynomials $M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\in\mathbb{V}$, for $i=1,2,\ldots,r$.

Now, aimed to get a contradiction, let us assume that there exist constants λ_i, $i=1,\ldots,r$, such that

$$\sum_{i=1}^{r}\lambda_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)=0, \qquad \text{where} \qquad \sum_{i=1}^{r}|\lambda_i|>0.$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)}\upsilon_q^{\beta_k+1,q\alpha}(s)\bigtriangledown x_1(s)$ and then taking summation on s from 0 to ∞, one gets

$$\sum_{i=1}^{r}\lambda_i\sum_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)[s]_q^{(n_k-1)}\upsilon_q^{\beta_k+1,q\alpha}(s)\bigtriangledown x_1(s)=0.$$

By Lemma 3.1.7 and $\sum\limits_{s=0}^{\infty}M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)[s]_q^{(n_k-1)}\upsilon_q^{\beta_k+1,q^r\alpha}(s)\bigtriangledown x_1(s)\neq 0$, we obtain

$$\sum_{i=1}^{r}\tilde{a}_{k,i}\lambda_i=0, \qquad k=1,\ldots,r,$$

which is equivalent to $\tilde{A}\lambda=0$, where $\tilde{a}_{k,i}$ are given in (3.1.20), $\lambda=[\lambda_1,\ldots,\lambda_r]^T$ and $\tilde{A}=\{\tilde{a}_{k,i}\}_{k,i=1}^{r}$. By Lemma 3.1.8, we have $|\tilde{A}|\neq 0$ so that $\lambda_i=0$ for $i=1,\ldots,r$. Therefore, $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$. Furthermore, we know that any polynomial of $\mathbb{V}$ can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}|-r)$ linear conditions are imposed on $\mathbb{V}$, consequently the dimension of $\mathbb{V}$ is at most r. Hence, the system $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\}_{i=1}^{r}$ spans $\mathbb{V}$, which completes the proof. □

Now we will prove that operator (1.3.4) is indeed a lowering operator for the sequence of q-Meixner multiple orthogonal polynomials of the second kind $M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$.

Lemma 3.1.10. *There holds the following relation*

$$\Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)=\sum_{i=1}^{r}\xi_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s), \tag{3.1.24}$$

where

$$\xi_i=\frac{\prod_{l=1}^{r}x(n_l+\beta_l-\beta_i)}{\prod_{k=1,k\neq i}^{r}x(\beta_i-\beta_k)\prod_{l=i+1}^{r}x(\beta_l-\beta_i)}\sum_{j=1}^{r}\frac{(1-\alpha q^{n_j+\beta_j})q^{|\vec{n}|-n_j+1/2}}{(1-\alpha q^{|\vec{n}|+\beta_j})x(n_j+\beta_j-\beta_i)}$$
$$\times\frac{(-1)^{i+j}\prod_{k=1}^{r}x(n_j+\beta_j-\beta_k)}{\prod_{k=1,k\neq j}^{r-1}q^{n_j}x(n_k-n_j+\beta_k-\beta_j)\prod_{l=j+1}^{r}q^{n_l}x(n_j-n_l+\beta_j-\beta_l)}. \tag{3.1.25}$$

Proof. Using summation by parts we have

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)[s]_q^{(k)} \upsilon_q^{\beta_j+1,q\alpha}(s) \bigtriangledown x_1(s) = -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) \nabla\left[[s]_q^{(k)} \upsilon_q^{\beta_j+1,q\alpha}(s)\right] \bigtriangledown x_1(s)$$
$$= -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\varphi_{j,k}(s)\upsilon_q^{\beta_j,\alpha}(s) \bigtriangledown x_1(s), \tag{3.1.26}$$

where

$$\varphi_{j,k}(s) = q^{1/2}\left(\frac{q^{\beta_j}x(s)}{x(\beta_j)} + 1\right)[s]_q^{(k)} - q^{-1/2}\frac{x(s)}{\alpha x(\beta_j)}[s-1]_q^{(k)},$$

is a polynomial of degree $\leq k+1$ in the variable $x(s)$. Consequently, from the orthogonality conditions (2.3.11) we get

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)[s]_q^{(k)} \upsilon_q^{\beta_j+1,q\alpha}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \leq k \leq n_j - 2, \qquad j = 1, \ldots, r.$$

Hence, from Lemma 3.1.9, $\Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) \in \mathbb{V}$. Moreover, $\Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$ can univocally be expressed as a linear combination of polynomials $\{M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s)\}_{i=1}^{r}$, i.e.

$$\Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = \sum_{i=1}^{r} \xi_i M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s), \qquad \sum_{i=1}^{r} |\xi_i| > 0. \tag{3.1.27}$$

Thus, for finding explicity $\xi_1, \ldots, \xi_r$ one takes into account Lemma 3.1.7 and (3.1.27) to get

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)[s]_q^{(n_k-1)} \upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s) = \left(\sum_{i=1}^{r} \xi_i \tilde{a}_{k,i}\right) \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}}^{\vec{\beta}+\vec{e},q^r\alpha}(s)[s]_q^{(n_k-1)}$$
$$\times\, \upsilon_q^{\beta_k+1,q^r\alpha}(s) \bigtriangledown x_1(s). \tag{3.1.28}$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of equation (3.1.26), then left-hand side of equation (3.1.28) transforms into relation

$$\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)[s]_q^{(n_k-1)} \upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s) = -\sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)\varphi_{k,n_k-1}(s)\upsilon_q^{\beta_k,\alpha}(s) \bigtriangledown x_1(s)$$
$$= \frac{q^{1/2}\left(1-\alpha_k q^{n_k+\beta_k}\right)}{\alpha q^{n_k+\beta_k}} \sum_{s=0}^{\infty} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)[s]_q^{(n_k)} \upsilon_q^{\beta_k+1,\alpha}(s) \bigtriangledown x_1(s).$$

Here we have used that $x(s)[s-1]_q^{(n_k-1)} = [s]_q^{(n_k)}$ to get

$$\varphi_{k,n_k-1}(s) = -\frac{q^{-1/2}\left(1-\alpha q^{n_k+\beta_k}\right)}{\alpha x(\beta_k)}[s]_q^{(n_k)} + \text{lower degree terms}.$$

By Lemma 3.1.7 we have

$$\begin{aligned}
&\sum_{s=0}^{\infty} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)[s]_q^{(n_k-1)}\upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s) \\
&\qquad = \frac{(1-\alpha q^{n_k+\beta_k})q^{|\vec{n}|-n_k+1/2}}{1-\alpha q^{|\vec{n}|+\beta_k}} x(n_k) \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}_k}^{\vec{\beta}+\vec{e}_k,q\alpha}(s)[s]_q^{(n_k-1)}\upsilon_q^{\beta_k+1,q\alpha}(s) \bigtriangledown x_1(s) \\
&\qquad = \frac{(1-\alpha q^{n_k+\beta_k})q^{|\vec{n}|-n_k+1/2}}{1-\alpha q^{|\vec{n}|+\beta_k}} x(n_k) a_{k,k} \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}}^{\vec{\beta}+\vec{e},q^r\alpha}(s)[s]_q^{(n_k-1)}\upsilon_q^{\beta_k+1,q^r\alpha}(s) \bigtriangledown x_1(s) \\
&\qquad = \tilde{b}_k \sum_{s=0}^{\infty} M_{q,\vec{n}-\vec{e}}^{\vec{\beta}+\vec{e},q^r\alpha}(s)[s]_q^{(n_k-1)}\upsilon_q^{\beta_k+1,q^r\alpha}(s) \bigtriangledown x_1(s),
\end{aligned} \tag{3.1.29}$$

where

$$\tilde{b}_k = \frac{q^{1/2}(1-\alpha q^{n_k+\beta_k})}{\alpha q^{n_k+\beta_k}} \prod_{i=1}^{r} \frac{\alpha q^{|\vec{n}|+\beta_i}}{1-\alpha q^{|\vec{n}|+\beta_i}} x(n_k+\beta_k-\beta_i).$$

From equation (3.1.28) and (3.1.29) we get the following linear system of equations for the unknown coefficients $\xi_1, \ldots, \xi_r$,

$$b_j = \sum_{i=1}^{r} \xi_i s_{j,i}, \quad k = 1, \ldots, r, \quad \Longleftrightarrow \quad S\xi = b, \quad \xi = (\xi_1, \ldots, \xi_r), \tag{3.1.30}$$

where the entries of the vector b and matrix S are as follows

$$b_j = \frac{(1-\alpha q^{n_j+\beta_j})q^{|\vec{n}|-n_j+1/2}}{(1-\alpha q^{|\vec{n}|+\beta_j})}, \quad s_{j,i} = a_{j,i}.$$

By using the Cramer's rule the above system (3.1.30) has a unique solution if and only if the determinant of S is different from zero. Observe that $S = A \cdot D$, where D denotes the diagonal matrix $D = (\delta_{j,i})_{j,i}^{r}$.

Thus, from Lemma 3.1.8, formula 3.1.22, one gets

$$\det S = \det(A \cdot D) = \det A \neq 0.$$

Accordingly, if $C_{j,i}$ is the cofactor of the entry $s_{j,i}$ and $S_i(b)$ denotes the matrix obtained from S replacing its ith column by b, then

$$\xi_i = \frac{det S_i(b)}{det S}, \quad i = 1, \ldots, r,$$

where by Lemma 3.1.8

$$\begin{aligned}\det S_i(b) &= \sum_{j=1}^{r} b_j C_{j,i} \\ &= \sum_{j=1}^{r} b_j (-1)^{i+j} \prod_{k=1,k\neq i}^{r-1} \prod_{l=k+1,l\neq i}^{r} x(\beta_l-\beta_k) \\ &\quad \times \frac{\prod_{k=1,k\neq j}^{r-1}\prod_{l=k+1,l\neq j}^{r} q^{n_l} x(n_k-n_l+\beta_k-\beta_l)}{\prod_{k=1,k\neq i}^{r}\prod_{l=1,l\neq j}^{r} x(n_k+\beta_k-\beta_l)}.\end{aligned}$$

Consequently, relation (3.1.24) holds. □

Theorem 3.1.11. *The q-Meixner multiple orthogonal polynomial of the second kind $M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$ satisfies the following $(r+1)$-order q-difference equation*

$$\prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = -\sum_{i=1}^{r} q^{1/2}\xi_i \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_j+1,q\alpha} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s), \tag{3.1.31}$$

where ξ_i's are the constants in (3.1.25).

Proof. Since the operators (2.3.12) commute, we write

$$\prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} = \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_j+1,q\alpha}\right) \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha}, \tag{3.1.32}$$

and use formula (2.3.12) in equation (3.1.24) by acting with the product of operators (3.1.32), we obtain the desired relation (3.1.31)

$$\begin{aligned}\prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) &= \sum_{i=1}^{r} \xi_i \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_j+1,q\alpha}\right) \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta}+\vec{e}_i,q\alpha}(s) \\ &= -\sum_{i=1}^{r} q^{1/2}\xi_i \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_j+1,q\alpha} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s).\end{aligned}$$

□

3.1.4 q-Kravchuk multiple orthogonal polynomials

In this section we will find a lowering operator for the q-Kravchuk multiple orthogonal polynomials. Then we will combine it with the raising operators (2.3.17) to get an $(r+1)$-order q-difference equation.

Lemma 3.1.12. *Let $\mathbb{V}$ be the linear subspace of polynomials $Q(s)$ on the lattice $x(s)$ of degree at most $|\vec{n}|-1$ defined by the following conditions*

$$\sum_{s=0}^{N} Q(s)[s]_q^{(k)} \upsilon_q^{p_j,N-1,\beta_j/q^2}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2 \qquad \textit{and} \qquad j = 1,\dots,r.$$

Then, the system $\{K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)\}_{i=1}^{r}$, where $\vec{\beta}_{i,1/q^2} = (\beta_1,\dots,\beta_i/q^2,\dots,\beta_r)$, is a basis for $\mathbb{V}$.

Proof. From orthogonality relations

$$\sum_{s=0}^{N} K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)[s]_q^{(k)} \upsilon_q^{p_j,N-1,\beta_j/q^2}(s) \bigtriangledown x_1(s) = 0, \qquad 0 \le k \le n_j - 2, \qquad j = 1,\dots,r,$$

we have that polynomials $K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)$, $i=1,\dots,r$, belong to $\mathbb{V}$.

Now, aimed to get a contradiction, let us assume that there exist constants λ_i, $i=1,\dots,r$, such that

$$\sum_{i=1}^{r} \lambda_i K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s) = 0, \qquad \text{where} \qquad \sum_{i=1}^{r} |\lambda_i| > 0.$$

Then, multiplying the previous equation by $[s]_q^{(n_k-1)} \upsilon_q^{p_k,N,\beta_k}(s) \bigtriangledown x_1(s)$ and taking summation on s from 0 to ∞, one gets

$$\sum_{i=1}^{r} \lambda_i \sum_{s=0}^{N} K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)[s]_q^{(n_k-1)} \upsilon_q^{p_k,N,\beta_k}(s) \bigtriangledown x_1(s) = 0.$$

Thus, taking into account relations

$$\sum_{s=0}^{N} K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)[s]_q^{(n_k-1)} \upsilon_q^{p_k,N,\beta_k}(s) \bigtriangledown x_1(s) = c\delta_{i,k}, \qquad c \in \mathbb{R} \setminus \{0\}, \tag{3.1.33}$$

we deduce that $\lambda_k = 0$ for $k = 1,\dots,r$. Here $\delta_{i,k}$ represents the Kronecker delta symbol. Therefore, $\{K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)\}_{i=1}^{r}$ is linearly independent in $\mathbb{V}$. Furthermore, we know that any polynomial of $\mathbb{V}$ can be determined with $|\vec{n}|$ coefficients while $(|\vec{n}|-r)$ linear conditions are imposed on $\mathbb{V}$. Consequently the dimension of $\mathbb{V}$ is at most r. Hence, the system $\{K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)\}_{i=1}^{r}$ spans $\mathbb{V}$, which completes the proof. □

Now we will prove that the operator (1.3.4) is indeed a lowering operator for the sequence of q-Kravchuk multiple orthogonal polynomials $K_{q,\vec{n}}^{\vec{p},N}(s)$.

Lemma 3.1.13. *There holds the following relation*

$$\Delta K_{q,\vec{n}}^{\vec{p},N}(s)=\sum_{i=1}^{r}q^{|\vec{n}|-n_i+1/2}\frac{[p_i\left(q^{n_i}-1\right)+1]}{[p_i\left(q^{|\vec{n}|}-1\right)+1]}[n_i]_q^{(1)}K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s). \tag{3.1.34}$$

Proof. Using summation by parts we have

$$\begin{aligned}\sum_{s=0}^{N}\Delta K_{q,\vec{n}}^{\vec{p},N}(s)[s]_q^{(k)}\upsilon_q^{p_j,N-1,\beta_j/q^2}(s)\bigtriangledown x_1(s)&=-\sum_{s=0}^{N}K_{q,\vec{n}}^{\vec{p},N}(s)\nabla\left[[s]_q^{(k)}\upsilon_q^{p_j,N-1,\beta_j/q^2}(s)\right]\bigtriangledown x_1(s)\\&=-\sum_{s=0}^{N}K_{q,\vec{n}}^{\vec{p},N}(s)\varphi_{j,k}(s)\upsilon_q^{p_j,N}(s)\bigtriangledown x_1(s),\end{aligned} \tag{3.1.35}$$

where

$$\varphi_{j,k}(s)=q^{1/2}\left(-\frac{q^{-2(N-1)}x(s)}{(1-p_j)[N]_q}+\frac{q^{-2(N-1)}x(N)}{(1-p_j)[N]_q}\right)[s]_q^{(k)}-q^{-1/2}\frac{q^{-2(N-1)}x(s)}{p_j[N]_q}[s-1]_q^{(k)},$$

is a polynomial of degree $\le k+1$ in the variable $x(s)$. Consequently, from the orthogonality conditions (2.3.16) we get

$$\sum_{s=0}^{N}\Delta K_{q,\vec{n}}^{\vec{p},N}(s)[s]_q^{(k)}\upsilon_q^{p_j,N-1,\beta_j/q^2}(s)\bigtriangledown x_1(s)=0,\qquad 0\le k\le n_j-2,\qquad j=1,\dots,r.$$

Hence, from Lemma 3.1.12, $\Delta K_{q,\vec{n}}^{\vec{p},N}(s)\in\mathbb{V}$. Moreover, $\Delta K_{q,\vec{n}}^{\vec{p},N}(s)$ can univocally be expressed as a linear combination of polynomials $\{K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)\}_{i=1}^{r}$, i.e.

$$\Delta K_{q,\vec{n}}^{\vec{p},N}(s)=\sum_{i=1}^{r}\xi_iK_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s),\qquad\sum_{i=1}^{r}|\xi_i|>0. \tag{3.1.36}$$

Multiplying both sides of the equation (3.1.36) by $[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s)$ and using relations (3.1.33) one has

$$\begin{aligned}&\sum_{s=0}^{N}\Delta K_{q,\vec{n}}^{\vec{p},N}(s)[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s)\\&\qquad=\sum_{i=1}^{r}\xi_i\sum_{s=0}^{N}K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s)\\&\qquad=\xi_k\sum_{s=0}^{N}K_{q,\vec{n}-\vec{e}_k}^{\vec{p},N-1,\vec{\beta}_{k,1/q^2}}(s)[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s).\end{aligned} \tag{3.1.37}$$

If we replace $[s]_q^{(k)}$ by $[s]_q^{(n_k-1)}$ in the left-hand side of equation (3.1.35), then left-hand side of equation (3.1.37) transforms into relation

$$\sum_{s=0}^{N} \Delta K_{q,\vec{n}}^{\vec{p},N}(s)[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s) = -\sum_{s=0}^{N} K_{q,\vec{n}}^{\vec{p},N}(s)\varphi_{k,n_k-1}(s)\upsilon_q^{p_k,N}(s)\bigtriangledown x_1(s)$$
$$= \frac{q^{-1/2}\left[p_k\left(q^{n_k}-1\right)+1\right]}{q^{2(N-1)}(1-p_k)[N]_q}\sum_{s=0}^{\infty} K_{q,\vec{n}}^{\vec{p},N}(s)[s]_q^{(n_k)}\upsilon_q^{p_k,N}(s)\bigtriangledown x_1(s). \quad (3.1.38)$$

Here we have used that $x(s)[s-1]_q^{(n_k-1)} = [s]_q^{(n_k)}$ to get

$$\varphi_{k,n_k-1}(s) = -\frac{q^{-1/2}\left[p_k\left(q^{n_k}-1\right)+1\right]}{q^{2(N-1)}p_k(1-p_k)[N]_q}[s]_q^{(n_k)} + \text{lower degree terms.}$$

On the other hand, from (2.3.17) one has that

$$\frac{\left[p_k\left(q^{|\vec{n}|}-1\right)+1\right]}{q^{2(N-1)}p_k(1-p_k)[N]_q}\upsilon_q^{p_k,N}(s)K_{q,\vec{n}}^{\vec{p},N}(s) = -q^{|\vec{n}|-1/2}\nabla\left[\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)K_{q,\vec{n}-\vec{e}_k}^{\vec{p},N-1,\vec{\beta}_{k,1/q^2}}(s)\right]. \quad (3.1.39)$$

Then, by conveniently substituting (3.1.39) in the right-hand side of equation (3.1.38) and using once more summation by parts, we get

$$\sum_{s=0}^{N} \Delta K_{q,\vec{n}}^{\vec{p},N}(s)[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s)$$
$$= -q^{|\vec{n}|-1}\frac{\left[p_k\left(q^{n_k}-1\right)+1\right]}{\left[p_k\left(q^{|\vec{n}|}-1\right)+1\right]}\sum_{s=0}^{N}[s]_q^{(n_k)}\nabla\left[\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)K_{q,\vec{n}-\vec{e}_k}^{\vec{p},N-1,\vec{\beta}_{k,1/q^2}}(s)\right]\bigtriangledown x_1(s)$$
$$= q^{|\vec{n}|-1}\frac{\left[p_k\left(q^{n_k}-1\right)+1\right]}{\left[p_k\left(q^{|\vec{n}|}-1\right)+1\right]}\sum_{s=0}^{N}K_{q,\vec{n}-\vec{e}_k}^{\vec{p},N-1,\vec{\beta}_{k,1/q^2}}(s)\Delta\left[[s]_q^{(n_k)}\right]\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s).$$

Since $\Delta[s]_q^{(n_k)} = q^{3/2-n_k}[n_k]_q^{(1)}[s]_q^{(n_k-1)}$ we finally have

$$\sum_{s=0}^{N} \Delta K_{q,\vec{n}}^{\vec{p},N}(s)[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s)$$
$$= q^{|\vec{n}|-n_k+1/2}\frac{\left[p_k\left(q^{n_k}-1\right)+1\right]}{\left[p_k\left(q^{|\vec{n}|}-1\right)+1\right]}[n_k]_q^{(1)}\sum_{s=0}^{N}K_{q,\vec{n}-\vec{e}_k}^{\vec{p},N-1,\vec{\beta}_{k,1/q^2}}(s)[s]_q^{(n_k-1)}\upsilon_q^{p_k,N-1,\beta_k/q^2}(s)\bigtriangledown x_1(s).$$

Therefore, comparing this equation with (3.1.37) we obtain the coefficients in the expansion (3.1.36), i.e.

$$\xi_k = q^{|\vec{n}|-n_k+1/2}\frac{\left[p_k\left(q^{n_k}-1\right)+1\right]}{\left[p_k\left(q^{|\vec{n}|}-1\right)+1\right]}[n_k]_q^{(1)},$$

which proves relation (3.1.34). $\square$

Theorem 3.1.14. *The q-Kravchuk multiple orthogonal polynomial $K_{q,\vec{n}}^{\vec{p},N}(s)$ satisfies the following $(r+1)$-order q-difference equation*

$$\prod_{i=1}^{r} \mathcal{D}_q^{p_i,N-1+r-i,\beta_i/q^2} \Delta K_{q,\vec{n}}^{\vec{p},N}(s) = -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1} \frac{[p_i(q^{n_i}-1)+1]}{[p_i(q^{|\vec{n}|}-1)+1]} [n_i]_q^{(1)} \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{p_j,N-1+r-j,\beta_j/q^2} K_{q,\vec{n}}^{\vec{p},N}(s). \quad (3.1.40)$$

Proof. Since operators (2.3.17) commute, we write

$$\prod_{i=1}^{r} \mathcal{D}_q^{p_i,N-1+r-i,\beta_i/q^2} = \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{p_j,N-1+r-j,\beta_j/q^2} \right) \mathcal{D}_q^{p_i,N-1,\beta_i/q^2}, \quad (3.1.41)$$

and then using (2.3.17), by acting on equation (3.1.34) with the product of operators (3.1.41), we obtain the following relation

$$\prod_{i=1}^{r} \mathcal{D}_q^{p_i,N-1+r-i,\beta_i/q^2} \Delta K_{q,\vec{n}}^{\vec{p},N}(s)$$
$$= \sum_{i=1}^{r} q^{|\vec{n}|-n_i+1/2} \frac{[p_i(q^{n_i}-1)+1]}{[p_i(q^{|\vec{n}|}-1)+1]} [n_i]_q^{(1)} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{p_j,N-1+r-j,\beta_j/q^2} \right) \mathcal{D}_q^{p_i,N-1,\beta_i/q^2} K_{q,\vec{n}-\vec{e}_i}^{\vec{p},N-1,\vec{\beta}_{i,1/q^2}}(s)$$
$$= -\sum_{i=1}^{r} q^{|\vec{n}|-n_i+1} \frac{[p_i(q^{n_i}-1)+1]}{[p_i(q^{|\vec{n}|}-1)+1]} [n_i]_q^{(1)} \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_q^{p_j,N-1+r-j,\beta_j/q^2} K_{q,\vec{n}}^{\vec{p},N}(s),$$

which proves (3.1.40). □

3.2 q-Recurrence relation

In this section we will study the nearest neighbor recurrence relation for any multi-index $\vec{n}$. Indeed, such a study constitutes an important contribution of this Thesis. Furthermore, the approach presented here differs form those used in [10, 41, 91].

Let us start by defining the linear difference operator $\mathcal{F}_{q,n_i}$, where n_i is the i-th entry of the vector index $\vec{n}$,

$$\mathcal{F}_{q,n_i} := g_{q,i}^{-1}(s) \nabla^{n_i} g_{q,k}(s), \quad (3.2.1)$$

where $g_{q,k}$ is defined in the variable s and depends on the i-th component of the vector orthogonality measure $\vec{\mu}$. Moreover, if it also depends on the i-th component of $\vec{n}$, then the index $k = n_i$; otherwise $k = i$.

Lemma 3.2.1. *Let n_i be a positive integer and let $f(s)$ be a function defined on the discrete variable s. The following relation is valid*

$$\mathcal{F}_{q,n_i}x(s)f_q(s) = q^{-n_i+1/2}x(n_i)g_{q,i}^{-1}(s)\,\nabla^{n_i-1}g_{q,k}(s)\,f_q(s) + q^{-n_i}\left[x(s)-x(n_i)\right]\mathcal{F}_{q,n_i}f_q(s). \quad (3.2.2)$$

Proof. Let us act n_i-times with backward difference operators (1.3.5) on the product of functions $x(s)f(s)$. Assume that $n_i \geq N > 1$,

$$\begin{aligned}\nabla^{n_i}x(s)f(s) &= \nabla^{n_i-1}[\nabla x(s)f(s)] = \nabla^{n_i-1}\big[q^{-1/2}f(s) + x(s-1)\nabla f(s)\big]\\ &= q^{-1/2}\nabla^{n_i-1}f(s) + \nabla^{n_i-1}[x(s-1)\nabla f(s)]\\ &= q^{-1/2}\nabla^{n_i-1}f(s) + \nabla^{n_i-2}[\nabla x(s-1)\nabla f(s)]. \end{aligned} \quad (3.2.3)$$

Repeating this process – but on the second term of the right-hand side of equation (3.2.3) –

$$\begin{aligned}\nabla^{n_i}x(s)f(s) &= \big[q^{1/2-n_i} + \cdots + q^{-5/2} + q^{-3/2} + q^{-1/2}\big]\nabla^{n_i-1}f(s) + x(s-n_i)\nabla^{n_i}f(s)\\ &= q^{1/2-n_i}x(n_i)\nabla^{n_i-1}f(s) + x(s-n_i)\nabla^{n_i}f(s),\end{aligned}$$

or, equivalently,

$$\nabla^{n_i}x(s)f(s) = q^{-n_i+1/2}x(n_i)\nabla^{n_i-1}f(s) + q^{-n_i}[x(s)-x(n_i)]\nabla^{n_i}f(s), \qquad n_i \geq 1. \quad (3.2.4)$$

Now, aimed to involve difference operator ∇^{n_i} in the above equation, we multiply equation (3.2.4) from the left side by $g_{q,i}(s)^{-1}$ and replace $f(s)$ by $g_{q,k}(s)f(s)$. Thus, equation (3.2.4) transforms into (3.2.2), which completes the proof. □

3.2.1 q-Charlier multiple orthogonal polynomials

In this section we will study two types of recurrence relations, namely the nearest neighbor recurrence relation for any multi-index $\vec{n}$ and a step-line recurrence relation for $\vec{n} = (n_1, n_2)$.

Theorem 3.2.2. *The q-Charlier multiple orthogonal polynomials satisfy the following $(r+2)$-term recurrence relation*

$$\begin{aligned}x(s)C_{q,\vec{n}}^{\vec{\alpha}}(s) = C_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}}(s) &+ \left(\sum_{i=1}^{r} q^{|\vec{n}|_i}x(n_i)A_{\vec{n},i} + \alpha_k q^{|\vec{n}|+n_k+1}\right)C_{q,\vec{n}}^{\vec{\alpha}}(s)\\ &+ q^{1/2}\sum_{i=1}^{r} x(n_i)\alpha_i q^{|\vec{n}|+n_i-1}B_{\vec{n},i}C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}}(s),\end{aligned} \quad (3.2.5)$$

where $|\vec{n}|_i = n_1 + \cdots + n_{i-1}$, $|\vec{n}|_1 = 0$, $A_{\vec{n},i} = (q-1)\left(\alpha_i q^{\sum_{j=i}^{r} n_j}\right) + 1$ *and* $B_{\vec{n},i} = \prod_{j\neq i}^{r} \frac{\alpha_i q^{|\vec{n}|} - \alpha_j q^{n_j}}{\alpha_i q^{n_i} - \alpha_j q^{n_j}} A_{\vec{n},i}$.

Proof. We will use Lemma 3.2.1. First, let us consider the equation

$$(\alpha_k)^{-s}\nabla^{n_k+1}(\alpha_k q^{n_k+1})^s\frac{1}{\Gamma_q(s+1)} = (\alpha_k)^{-s}\nabla^{n_k}\left[q^{-s+1/2}\triangledown\frac{(\alpha_k q^{n_k+1})^s}{\Gamma_q(s+1)}\right]$$
$$= q^{1/2}(\alpha_k)^{-s}\nabla^{n_k}(\alpha_k q^{n_k})^s\frac{1}{\Gamma_q(s+1)} - (\alpha_k q^{n_k+1})^{-1}q^{1/2}(\alpha_k)^{-s}\nabla^{n_k}(\alpha_k q^{n_k})^s x(s)\frac{1}{\Gamma_q(s+1)},$$

which can be rewritten in terms of difference operators (2.3.6) as follows

$$\mathcal{C}^{\alpha_k}_{q,n_k+1}\frac{1}{\Gamma_q(s+1)} = q^{1/2}\mathcal{C}^{\alpha_k}_{q,n_k}\frac{1}{\Gamma_q(s+1)} - (\alpha_k q^{n_k+1})^{-1}q^{1/2}\mathcal{C}^{\alpha_k}_{q,n_k}x(s)\frac{1}{\Gamma_q(s+1)}. \tag{3.2.6}$$

Since operators (2.3.6) commute the multiplication of equation (3.2.6) from the left-hand side by the product $\left(\prod_{\substack{i=1\\i\neq k}}^{r}\mathcal{C}^{\alpha_i}_{q,n_i}\right)$ yields

$$\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}x(s)\frac{1}{\Gamma_q(s+1)} = \alpha_k q^{n_k+1}\big(\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}} - q^{-1/2}\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}+\vec{e}_k}\big)\frac{1}{\Gamma_q(s+1)}. \tag{3.2.7}$$

Second, let us recursively use expression (3.2.2) involving the product of r difference operators $\prod_{i=1}^{r}\mathcal{F}_{q,n_i}$ acting on the function $f(s)=1/\Gamma_q(s+1)$, which in this case is $\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}$ (see expression (2.3.6)). Thus, we have

$$q^{|\vec{n}|}\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}x(s)f(s) = q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}x(n_i)\Big[(q-1)\alpha_i q^{n_i}+1\Big]\prod_{l=1}^{r}\mathcal{C}^{\alpha_j}_{q,n_l-\delta_{l,i}}f(s)$$
$$+\left(x(s)-\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)\left[(q-1)\left(\alpha_i q^{\sum_{j=i}^{r}n_j}\right)+1\right]\right)\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}f(s). \tag{3.2.8}$$

Hence, by using expressions (3.2.7), (3.2.8) one gets

$$x(s)\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}f(s) = -\alpha_k q^{|\vec{n}|+n_k+1}q^{-1/2}\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}+\vec{e}_k}f(s)$$
$$+\left(\sum_{i=1}^{r}q^{|\vec{n}|_i}x(n_i)\left[(q-1)\left(\alpha_i q^{\sum_{j=i}^{r}n_j}\right)+1\right]+\alpha_k q^{|\vec{n}|+n_k+1}\right)\mathcal{C}^{\vec{\alpha}}_{q,\vec{n}}f(s)$$
$$-q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}x(n_i)\Big[(q-1)\alpha_i q^{n_i}+1\Big]\prod_{l=1}^{r}\mathcal{C}^{\alpha_l}_{q,n_l-\delta_{l,i}}f(s).$$

Finally, multiplying from the left both sides of the previous expression by $\mathcal{G}^{\vec{n},\vec{\alpha}}_q\Gamma_q(s+1)$ and using Rodrigues-type formula (2.3.5) we obtain (3.2.5), which completes the proof. $\square$

Observe that other recurrence relations different from the above nearest neighbor recurrence relation (3.2.5) can be obtained. Indeed, from (3.2.13) a 4-term recurrence relation for $\vec{n} = (n_1, n_2)$ can be obtained. In [10] an approach for the recurrence relations of some discrete multiple orthogonal polynomials was given. This approach uses the Rodrigues-type formulas along with some series representations (Kampé de Fériet series) for multiple orthogonal polynomials. Here we proceed in the same way.

Notice that equation (2.3.5) provides an explicit expression for the monic q-Charlier multiple orthogonal polynomials. Indeed, by using formula (3.2.29) from [73] as follows

$$\nabla^m f(s) = q^{\binom{m+1}{2}/2-ms} \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix} (-1)^k q^{\binom{m-k}{2}} f(s-k),$$
$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}}, \qquad m = 1, 2, \ldots,$$

where

$$(a;q)_k = \prod_{j=0}^{k-1} (1 - aq^j) \qquad \text{for} \quad k > 0, \qquad \text{and} \qquad (a;q)_0 = 1,$$

denotes the q-analogue of the Pochhammer symbol [46, 61, 73, 75], one obtains the following relation for the q-Charlier multiple orthogonal polynomials (for multi-index $\vec{n} = (n_1, n_2)$)

$$\begin{aligned} C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2-(n_1+n_2)/2} \left(\frac{\Gamma_q(s+1)}{\alpha_2^s} \nabla^{n_2} (\alpha_2 q^{n_2})^s \right) \left(\frac{1}{\alpha_1^s} \nabla^{n_1} \frac{(\alpha_1 q^{n_1})^s}{\Gamma_q(s+1)} \right) \\ &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2-(n_1+n_2-\binom{n_1+1}{2}-\binom{n_2+1}{2})/2} \\ &\times \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} (-1)^{l+k} \begin{bmatrix} n_2 \\ l \end{bmatrix} \begin{bmatrix} n_1 \\ k \end{bmatrix} \frac{q^{\binom{n_2-l}{2}-ln_2+\binom{n_1-k}{2}-kn_1}}{\alpha_2^l \alpha_1^k} \frac{\Gamma_q(s+1)}{\Gamma_q(s-k-l+1)}. \end{aligned} \tag{3.2.9}$$

Observe that from (1.3.2) we have

$$\frac{\Gamma_q(s+1)}{\Gamma_q(s-l-k+1)} = [s]_q^{(k+l)} = \frac{(q^{-s};q)_{l+k}}{(q-1)^{l+k}} q^{s(k+l)-\binom{k+l}{2}}.$$

Now, using relation

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2}-nk}, \qquad k = 0, 1, \ldots,$$

(see [61, formula (1.8.18)]) the above expression (3.2.9) becomes

$$\begin{aligned} C_{q,n_1,n_2}^{\alpha_1,\alpha_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\times \sum_{k=0}^{n_1} \sum_{l=0}^{n_2} \frac{(q^{-s};q)_{k+l}(q^{-n_1};q)_k(q^{-n_2};q)_l}{q^{\binom{k+l}{2}}(q;q)_k(q;q)_l} \left(\frac{q^{s+1-n_1}}{\alpha_1(q-1)} \right)^k \left(\frac{q^{s+1-n_2}}{\alpha_2(q-1)} \right)^l. \end{aligned} \tag{3.2.10}$$

Finally, from (3.2.10) we have

$$\begin{aligned} C^{\alpha_1,\alpha_2}_{q,n_1,n_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\times \lim_{\gamma\to+\infty} \Phi_2\left(q^{-s}; q^{-n_1}, q^{-n_2}; \gamma, \gamma; \frac{\gamma q^{s+1-n_1}}{\alpha_1(1-q)}, \frac{\gamma q^{s+1-n_2}}{\alpha_2(1-q)}\right), \end{aligned} \tag{3.2.11}$$

where

$$\Phi_2(\zeta;\beta,\beta';\gamma,\gamma';x,y) = \sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty} \frac{(\zeta;q)_{m+n}(\beta;q)_m(\beta';q)_n}{(\gamma;q)_m(\gamma';q)_n(q;q)_m(q;q)_n} q^{-mn}x^m y^n,$$

is a q-analogue of the second of Appell's hypergeometric functions of two variables (see [39, formula (23), p. 89] and [89] for multiple gaussian hypergeometric series).

Alternatively, in (3.2.10) the q-analogue of the Pochhammer symbol can be rewritten in terms of the q-falling factorials, which allows to express the q-Charlier multiple orthogonal polynomials in terms of the selected basis $[s]_q^{(k)}$, $k = 0, 1, \ldots$, i.e.

$$\begin{aligned} C^{\alpha_1,\alpha_2}_{q,n_1,n_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\times \sum_{k=0}^{n_1}\sum_{l=0}^{n_2} \frac{[s]_q^{(k+l)}[n_1]_q^{(k)}[n_2]_q^{(l)}}{[k]_q![l]_q!} q^{\binom{k+1}{2}+\binom{l+1}{2}} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^k \left(\frac{-1}{q^{2n_2}\alpha_2}\right)^l. \end{aligned} \tag{3.2.12}$$

If we replace k by $j-l$ in (3.2.12), we obtain

$$\begin{aligned} C^{\alpha_1,\alpha_2}_{q,n_1,n_2}(s) &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\times \sum_{l=0}^{n_2}\sum_{j=l}^{l+n_1} \frac{[s]_q^{(j)}[n_1]_q^{(j-l)}[n_2]_q^{(l)}}{[j-l]_q![l]_q!} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^{j-l} \left(\frac{-1}{q^{2n_2}\alpha_2}\right)^l q^{\binom{j-l+1}{2}+\binom{l+1}{2}} \\ &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\times \sum_{j=0}^{n_1+n_2}\sum_{l=\max(0,j-n_1)}^{\min(j,n_2)} \frac{[s]_q^{(j)}[n_1]_q^{(j-l)}[n_2]_q^{(l)}}{[j-l]_q![l]_q!} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^{j-l} \left(\frac{-1}{q^{2n_2}\alpha_2}\right)^l q^{\binom{j-l+1}{2}+\binom{l+1}{2}} \\ &= (-\alpha_1)^{n_1}(-\alpha_2)^{n_2} q^{n_1^2+n_2^2+n_1n_2+\frac{3}{2}\left(\binom{n_1}{2}+\binom{n_2}{2}\right)} \\ &\times \sum_{j=0}^{n_1+n_2}\sum_{l=0}^{j} \frac{[s]_q^{(j)}[n_1]_q^{(j-l)}[n_2]_q^{(l)}}{[j-l]_q![l]_q!} \left(\frac{-1}{q^{2n_1}\alpha_1}\right)^{j-l} \left(\frac{-1}{q^{-2n_2}\alpha_2}\right)^l q^{\binom{j-l+1}{2}+\binom{l+1}{2}}. \end{aligned} \tag{3.2.13}$$

This expression is useful for computing some recurrence coefficients.

Considering the expansion

$$C^{\alpha_1,\alpha_2}_{q,n_1,n_2}(s) = \sum_{j=0}^{n_1+n_2} c^{(j)}_{q,n_1,n_2}[s]_q^{(j)},$$

the coefficients $c^{(j)}_{q,n_1,n_2}$ can be used to compute the recurrence coefficients in

$$\begin{aligned} x(s)P_{q,n_1,n_2}(s) &= q^{n_1+n_2}P_{q,n_1,n_2+1}(s) + b_{q,n_1,n_2}P_{q,n_1,n_2}(s) + c_{q,n_1,n_2}P_{q,n_1,n_2-1}(s) \\ &\quad + d_{q,n_1,n_2}P_{q,n_1-1,n_2-1}(s), \end{aligned}$$

where $P_{q,n_1,n_2}(s) = C^{\alpha_1,\alpha_2}_{q,n_1,n_2}(s)$. Indeed,

$$\begin{aligned} b_{q,n_1,n_2} &= q^{n_1+n_2}\big(q^{-1}c^{(n_1+n_2-1)}_{q,n_1,n_2} - c^{(n_1+n_2)}_{q,n_1,n_2+1}\big) + x(n_1+n_2), \\ c_{q,n_1,n_2} &= q^{n_1+n_2}\big(q^{-2}c^{(n_1+n_2-2)}_{q,n_1,n_2} + q^{-(n_1+n_2)}c^{(n_1+n_2-1)}_{q,n_1,n_2}x(n_1+n_2-1) \\ &\quad - q^{-(n_1+n_2)}b_{q,n_1,n_2}c^{(n_1+n_2-1)}_{q,n_1,n_2} - c^{(n_1+n_2-1)}_{q,n_1,n_2+1}\big), \\ d_{q,n_1,n_2} &= q^{n_1+n_2}\big(q^{-3}c^{(n_1+n_2-3)}_{q,n_1,n_2} + q^{-(n_1+n_2)}c^{(n_1+n_2-2)}_{q,n_1,n_2}x(n_1+n_2-2) \\ &\quad - q^{-(n_1+n_2)}c_{q,n_1,n_2}c^{(n_1+n_2-2)}_{q,n_1,n_2-1} - q^{-(n_1+n_2)}b_{q,n_1,n_2}c^{(n_1+n_2-2)}_{q,n_1,n_2} - c^{(n_1+n_2-2)}_{q,n_1,n_2+1}\big). \end{aligned}$$

From the explicit expression (3.2.13) we then get, after some calculations, that when $q \to 1$ we recover the recurrence coefficients given in [10].

3.2.2 q-Meixner multiple orthogonal polynomials of the first kind

Theorem 3.2.3. *The q-Meixner multiple orthogonal polynomials of the first kind satisfy the following $(r+2)$-term recurrence relation*

$$\begin{aligned} x(s)M^{\vec{\alpha},\beta}_{q,\vec{n}}(s) &= M^{\vec{\alpha},\beta}_{q,\vec{n}+\vec{e}_k}(s) + b_{\vec{n},k}M^{\vec{\alpha},\beta}_{q,\vec{n}}(s) \\ &\quad + \sum_{i=1}^{r}\frac{x(n_i)\alpha_i q^{|\vec{n}|+n_i-1}x(\beta+|\vec{n}|-1)}{\left(\alpha_i q^{|\vec{n}|+\beta+n_i-1}-1\right)\left(\alpha_i q^{|\vec{n}|+\beta+n_i-2}-1\right)}B_{\vec{n},i}M^{\vec{\alpha},\beta}_{q,\vec{n}-\vec{e}_i}(s), \end{aligned} \tag{3.2.14}$$

where

$$\begin{aligned} b_{\vec{n},k} &= q^{|\vec{n}|}x(\beta+|\vec{n}|)\frac{\alpha_k q^{n_k+1}}{1-\alpha_k q^{|\vec{n}|+\beta+n_k+1}} + \sum_{i=1}^{r} q^{|\vec{n}|}x(n_i)\left(\frac{\alpha_i q^{n_i}-1}{\alpha_i q^{|\vec{n}|+\beta}-1}\prod_{i=1}^{r}\frac{\alpha_i q^{|\vec{n}|+\beta}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\right. \\ &\quad \left. -\frac{\alpha_i q^{|\vec{n}|+\beta+n_i-1}}{\alpha_i q^{|\vec{n}|+\beta+n_i-1}-1}\frac{\alpha_i q^{n_i}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\right) \\ &\quad + (q-1)\prod_{i=1}^{r}\frac{x(n_i)}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\Big(q^{|\vec{n}|+\beta}\prod_{i=1}^{r}\alpha_i q^{n_i}-1\Big) \end{aligned}$$

and

$$B_{\vec{n},i} = \frac{\alpha_i q^{n_i}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\prod_{i=1}^{r}\frac{\alpha_i q^{|\vec{n}|+\beta-1}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}.$$

Proof. We will use Lemma 3.2.1. First, let us consider equation

$$
\begin{aligned}
&(\alpha_k)^{-s}\nabla^{n_k+1}(\alpha_k q^{n_k+1})^s \frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)} \\
&\qquad = (\alpha_k)^{-s}\nabla^{n_k}\left[q^{-s+1/2}\,\triangledown\left((\alpha_k q^{n_k+1})^s \frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)}\right)\right] \\
&\qquad = q^{1/2}(\alpha_k)^{-s}\nabla^{n_k}\left[(\alpha_k q^{n_k})^s\left(1+\frac{\alpha_k q^{|\vec{n}|+\beta+n_k+1}-1}{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}x(s)\right)\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}\right],
\end{aligned}
$$

which can be rewritten in terms of difference operators (2.3.10) as follows

$$
\begin{aligned}
\mathcal{M}^{\alpha_k}_{q,n_k+1}\frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)} &= q^{1/2}\mathcal{M}^{\alpha_k}_{q,n_k}\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} \\
&\quad + q^{1/2}\frac{\alpha_k q^{|\vec{n}|+\beta+n_k+1}-1}{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}\mathcal{M}^{\alpha_k}_{q,n_k}x(s)\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}. \quad (3.2.15)
\end{aligned}
$$

Since operators (2.3.10) commute, the multiplication of equation (3.2.15) from the left-hand side by the product $\left(\prod_{\substack{i=1\\ i\neq k}}^{r}\mathcal{M}^{\alpha_i}_{q,n_i}\right)$ yields

$$
\begin{aligned}
\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}}x(s)\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)} &= q^{-1/2}\frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{\alpha_k q^{|\vec{n}|+\beta+n_k+1}-1}\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}+\vec{e}_k}\frac{\Gamma_q(\beta+|\vec{n}|+1+s)}{\Gamma_q(\beta+|\vec{n}|+1)\Gamma_q(s+1)} \\
&\quad - \frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{\alpha_k q^{|\vec{n}|+\beta+n_k+1}-1}\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}}\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}. \quad (3.2.16)
\end{aligned}
$$

Second, let us recursively use Lemma 3.2.1 involving the product of r difference operators $\prod_{i=1}^{r}\mathcal{F}_{q,n_i}$ acting on the function $f_{|\vec{n}|}(s)=\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}$, which in this case is $\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}}$ (see expression (2.3.10)). Thus, we have

$$
\begin{aligned}
q^{|\vec{n}|}\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}}x(s)f_{|\vec{n}|}(s) &= q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\frac{x(n_i)\big(\alpha_i q^{n_i}-1\big)\big(\alpha_j q^{|\vec{n}|+\beta+n_j}-1\big)}{\prod_{\nu=1}^{r}\big(\alpha_\nu q^{|\vec{n}|+\beta}-1\big)} \\
&\times\prod_{l=1}^{r}\mathcal{M}^{\alpha_l}_{q,n_l-\delta_{l,i}}f_{|\vec{n}|}(s)+\left(\prod_{i=1}^{r}\frac{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}{\alpha_i q^{|\vec{n}|+\beta}-1}x(s)-\sum_{i=1}^{r}\frac{q^{|\vec{n}|}x(n_i)\left(\alpha_i q^{n_j}-1\right)}{\left(\alpha_i q^{|\vec{n}|+\beta}-1\right)}\right. \\
&\left.-\big(q-1\big)q^{|\vec{n}|+\beta}\prod_{i=1}^{r}\frac{x(n_i)\alpha_i q^{n_i}}{\alpha_i q^{|\vec{n}|+\beta}-1}+\big(q-1\big)\prod_{i=1}^{r}\frac{x(n_i)}{\alpha_i q^{|\vec{n}|+\beta}-1}\right)\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}}f_{|\vec{n}|}(s). \quad (3.2.17)
\end{aligned}
$$

Hence, by using expressions (3.2.16), (3.2.17) one gets

$$
\begin{aligned}
x(s)\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}} f_{|\vec{n}|}(s) &= q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{\alpha_i q^{|\vec{n}|+\beta}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{\alpha_k q^{|\vec{n}|+\beta+n_k+1}-1}\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}+\vec{e}_k} f_{|\vec{n}|+1}(s) \\
&+\prod_{i=1}^{r}\frac{\alpha_i q^{|\vec{n}|+\beta}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\Bigg(\sum_{i=1}^{r}\frac{q^{|\vec{n}|}x(n_i)\left(\alpha_i q^{n_j}-1\right)}{\left(\alpha_i q^{|\vec{n}|+\beta}-1\right)}+\big(q-1\big)q^{|\vec{n}|+\beta}\prod_{i=1}^{r}\frac{x(n_i)\alpha_i q^{n_i}}{\alpha_i q^{|\vec{n}|+\beta}-1} \\
&-\big(q-1\big)\prod_{i=1}^{r}\frac{x(n_i)}{\alpha_i q^{|\vec{n}|+\beta}-1}+q^{|\vec{n}|}(\alpha_k q^{n_k+1})\frac{x(\beta+|\vec{n}|)}{1-\alpha_k q^{|\vec{n}|+\beta+n_k+1}}\Bigg)\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}} f_{|\vec{n}|}(s) \\
&-q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\frac{x(n_i)\big(\alpha_i q^{n_i}-1\big)}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\prod_{l=1}^{r}\mathcal{M}^{\alpha_l}_{q,n_l-\delta_{l,i}} f_{|\vec{n}|}(s).
\end{aligned}
$$

Observe that

$$
\mathcal{M}^{\alpha_l}_{q,n_l-\delta_{l,i}} f_{|\vec{n}|}\left(s\right)\Big|_{l=i} = q^{-1/2}\frac{\alpha_i q^{\beta+|\vec{n}|+n_i-1}}{\alpha_i q^{\beta+|\vec{n}|+n_i-1}-1}\mathcal{M}^{\alpha_i}_{q,n_i} f_{|\vec{n}|}\left(s\right)-\frac{1}{\alpha_i q^{\beta+|\vec{n}|+n_i-1}-1}\mathcal{M}^{\alpha_i}_{q,n_i-1} f_{|\vec{n}|-1}\left(s\right).
$$

Consequently,

$$
\begin{aligned}
&x(s)\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}} f_{|\vec{n}|}\left(s\right) \\
&= q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{\alpha_i q^{|\vec{n}|+\beta}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\frac{(\alpha_k q^{n_k+1})x(\beta+|\vec{n}|)}{\alpha_k q^{|\vec{n}|+\beta+n_k+1}-1}\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}+\vec{e}_k} f_{|\vec{n}|+1}\left(s\right)+b_{\vec{n},k}\mathcal{M}^{\vec{\alpha}}_{q,\vec{n}} f_{|\vec{n}|}\left(s\right) \\
&\quad -q^{1/2}\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{\alpha_i q^{|\vec{n}|}-\alpha_j q^{n_j}}{\alpha_i q^{n_i}-\alpha_j q^{n_j}}\frac{x(n_i)\big(\alpha_i q^{n_i}-1\big)}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\frac{1}{\alpha_i q^{|\vec{n}|+\beta+n_i-1}-1}\prod_{l=1}^{r}\mathcal{M}^{\alpha_l}_{q,n_l-\delta_{l,i}} f_{|\vec{n}|-1}\left(s\right).
\end{aligned}
$$

Finally, multiplying from the left both sides of the previous expression by $\mathcal{G}_q^{\vec{n},\vec{\alpha},\beta}\frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)}$ and using Rodrigues-type formula (2.3.9) we obtain (3.2.14), which completes the proof. □

3.2.3 q-Meixner multiple orthogonal polynomials of the second kind

Theorem 3.2.4. *The q-Meixner multiple orthogonal polynomials of the second kind satisfy the following $(r+2)$-term recurrence relation*

$$
\begin{aligned}
x(s)M^{\vec{\beta},\alpha}_{q,\vec{n}}(s) &= M^{\vec{\beta},\alpha}_{q,\vec{n}+\vec{e}_k}(s)+b_{\vec{n},k}M^{\vec{\beta},\alpha}_{q,\vec{n}}(s) \\
&+q^{|\vec{n}|}(\alpha q^{|\vec{n}|-1})\sum_{i=1}^{r}\frac{x(n_i)x(\beta_i+n_i-1)}{\left(1-\alpha q^{|\vec{n}|+\beta_i+n_i-1}\right)\left(1-\alpha q^{|\vec{n}|+\beta_i+n_i-2}\right)} \\
&\times\prod_{j\neq i}^{r}\frac{x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}B_{\vec{n},i}M^{\vec{\beta},\alpha}_{q,\vec{n}-\vec{e}_i}(s), \quad (3.2.18)
\end{aligned}
$$

where

$$b_{\vec{n},k} = \prod_{i=1}^{r} \frac{\alpha q^{|\vec{n}|+\beta_i} - 1}{\alpha q^{|\vec{n}|+\beta_i+n_i} - 1} \left(\frac{(\alpha q^{2|\vec{n}|+1}) x(\beta_k + n_k)}{1 - \alpha q^{|\vec{n}|+\beta_k+n_k+1}} + \sum_{i=1}^{r} \frac{q^{|\vec{n}|} x(n_i)}{q^{n_i}(1 - \alpha q^{|\vec{n}|+\beta_i})} \right)$$
$$+ (q-1) \left(\alpha q^{|\vec{n}|} \sum_{i=1}^{r} \frac{x(n_i) x(n_i + \beta_i - 1)}{(1 - \alpha q^{|\vec{n}|+\beta_i+n_i})(1 - \alpha q^{|\vec{n}|+\beta_i+n_i-1})} \prod_{j \neq i}^{r} \frac{x(n_i + \beta_i - \beta_j)}{x(n_i + \beta_i - n_j - \beta_j)} \right.$$
$$\left. + \prod_{i=1}^{r} \frac{x(n_i)}{1 - \alpha q^{|\vec{n}|+\beta_i}} \prod_{i=1}^{r} \frac{\alpha q^{|\vec{n}|+\beta_i} - 1}{\alpha q^{|\vec{n}|+\beta_i+n_i} - 1} \right)$$

and

$$B_{\vec{n},i} = \frac{1 - \alpha q^{|\vec{n}|}}{1 - \alpha q^{|\vec{n}|+\beta_i+n_i}} \prod_{i=1}^{r} \frac{\alpha q^{|\vec{n}|+\beta_i-1} - 1}{\alpha q^{|\vec{n}|+\beta_i+n_i} - 1}.$$

Proof. First, let us consider equation

$$\frac{\Gamma_q(\beta_k)}{\Gamma_q(\beta_k + s)} \nabla^{n_k+1} \frac{\Gamma_q(\beta_k + n_k + 1 + s)}{\Gamma_q(\beta_k + n_k + 1)} \frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)}$$
$$= \frac{\Gamma_q(\beta_k)}{\Gamma_q(\beta_k + s)} \nabla^{n_k} \left[q^{-s+1/2} \bigtriangledown \left(\frac{\Gamma_q(\beta_k + n_k + 1 + s)}{\Gamma_q(\beta_k + n_k + 1)} \frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)} \right) \right]$$
$$= q^{1/2} \frac{\Gamma_q(\beta_k)}{\Gamma_q(\beta_k + s)} \nabla^{n_k} \left[\frac{\Gamma_q(\beta_k + n_k + s)}{\Gamma_q(\beta_k + n_k)} \left(1 + \frac{\alpha q^{|\vec{n}|+\beta_k+n_k+1} - 1}{(\alpha_k q^{|\vec{n}|+1}) x(\beta_k + n_k)} x(s) \right) \frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)} \right],$$

which can be rewritten in terms of difference operators (2.3.15) as follows

$$\bar{\mathcal{M}}_{q,n_k+1}^{\beta_k} \frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)} = q^{1/2} \bar{\mathcal{M}}_{q,n_k}^{\beta_k} \frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)} + q^{1/2} \frac{\alpha q^{|\vec{n}|+\beta_k+n_k+1} - 1}{(\alpha q^{|\vec{n}|+1}) x(\beta_k + n_k)} \bar{\mathcal{M}}_{q,n_k}^{\beta_k} x(s) \frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}. \quad (3.2.19)$$

Since operators (2.3.15) commute, the multiplication of equation (3.2.19) from the left-hand side by the product $\left(\prod_{\substack{i=1 \\ i \neq k}}^{r} \bar{\mathcal{M}}_{q,n_i}^{\beta_i} \right)$ yields

$$\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}} x(s) \frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)} = q^{-1/2} \frac{(\alpha q^{|\vec{n}|+1}) x(\beta_k + n_k)}{\alpha q^{|\vec{n}|+\beta_k+n_k+1} - 1} \bar{\mathcal{M}}_{q,\vec{n}+\vec{e}_k}^{\vec{\beta}} \frac{(\alpha q^{|\vec{n}|+1})^s}{\Gamma_q(s+1)}$$
$$- \frac{(\alpha q^{|\vec{n}|+1}) x(\beta_k + n_k)}{\alpha q^{|\vec{n}|+\beta_k+n_k+1} - 1} \bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}} \frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}. \quad (3.2.20)$$

Second, let us recursively use Lemma 3.2.1 involving the product of r difference operators $\prod_{i=1}^{r} \mathcal{F}_{q,n_i}$ acting on the function $f_{|\vec{n}|}(s) = \frac{(\alpha q^{|\vec{n}|})^s}{\Gamma_q(s+1)}$, which in this case is $\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}$ (see expression (2.3.15)).

Thus, we have

$$
\begin{aligned}
q^{|\vec{n}|}\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}x(s)f_{|\vec{n}|}(s) = &\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{q^{1/2}x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}\frac{x(n_i)\big(\alpha q^{|\vec{n}|+\beta_j+n_j}-1\big)}{\prod\limits_{\nu=1}^{r}\big(\alpha q^{|\vec{n}|+\beta_\nu}-1\big)}\prod_{l=1}^{r}\bar{\mathcal{M}}_{q,n_l-\delta_{l,i}}^{\beta_l}f_{|\vec{n}|}(s)\\
&+\left(q^{|\vec{n}|}\sum_{i=1}^{r}\frac{x(n_i)}{q^{n_i}(\alpha q^{|\vec{n}|+\beta_i}-1)}+\big(q-1\big)\prod_{i=1}^{r}\frac{x(n_i)}{\alpha q^{|\vec{n}|+\beta_i}-1}\right)\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}f_{|\vec{n}|}(s)\\
&+\prod_{i=1}^{r}\frac{\alpha q^{|\vec{n}|+\beta_i+n_i}-1}{\alpha q^{|\vec{n}|+\beta_i}-1}x(s)\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}f_{|\vec{n}|}(s). \quad (3.2.21)
\end{aligned}
$$

Hence, by using expressions (3.2.20), (3.2.21) one gets

$$
\begin{aligned}
x(s)\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}f_{|\vec{n}|}(s) = &\, q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{\alpha q^{|\vec{n}|+\beta_i}-1}{\alpha q^{|\vec{n}|+\beta_i+n_i}-1}\frac{(\alpha q^{|\vec{n}|+1})x(\beta_k+n_k)}{\alpha q^{|\vec{n}|+\beta_k+n_k+1}-1}\bar{\mathcal{M}}_{q,\vec{n}+\vec{e}_k}^{\vec{\beta}}f_{|\vec{n}|+1}(s)\\
&+\prod_{i=1}^{r}\frac{\alpha_i q^{|\vec{n}|+\beta}-1}{\alpha_i q^{|\vec{n}|+\beta+n_i}-1}\left(q^{|\vec{n}|}\sum_{i=1}^{r}\frac{x(n_i)}{q^{n_i}(\alpha q^{|\vec{n}|+\beta_i}-1)}+\big(q-1\big)\prod_{i=1}^{r}\frac{x(n_i)}{\alpha q^{|\vec{n}|+\beta_i}-1}\right.\\
&\left.+q^{|\vec{n}|}\frac{(\alpha q^{|\vec{n}|+1})x(\beta_k+n_k)}{1-\alpha q^{|\vec{n}|+\beta_k+n_k+1}}\right)\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}f_{|\vec{n}|}(s)-\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{q^{1/2}x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}\frac{x(n_i)}{1-\alpha q^{|\vec{n}|+\beta_i+n_i}}\\
&\times\prod_{l=1}^{r}\bar{\mathcal{M}}_{q,n_l-\delta_{l,i}}^{\beta_l}f_{|\vec{n}|}(s).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\bar{\mathcal{M}}_{q,n_l-\delta_{l,i}}^{\beta_l}f_{|\vec{n}|}\left(s\right)\Big|_{l=i} = &\, q^{-1/2}\frac{(q-1)\alpha q^{|\vec{n}|}x(n_i+\beta_i-1)}{\alpha q^{\beta_i+|\vec{n}|+n_i-1}-1}\bar{\mathcal{M}}_{q,n_i}^{\beta_i}f_{|\vec{n}|}\left(s\right)\\
&+\frac{\alpha q^{|\vec{n}|}-1}{\alpha q^{\beta_i+|\vec{n}|+n_i-1}-1}\bar{\mathcal{M}}_{q,n_i-1}^{\beta_i}f_{|\vec{n}|-1}\left(s\right).
\end{aligned}
$$

Consequently

$$
\begin{aligned}
x(s)\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}f_{|\vec{n}|}(s) = &\, q^{|\vec{n}|-1/2}\prod_{i=1}^{r}\frac{\alpha q^{|\vec{n}|+\beta_i}-1}{\alpha q^{|\vec{n}|+\beta_i+n_i}-1}\frac{(\alpha q^{|\vec{n}|+1})x(\beta_k+n_k)}{\alpha q^{|\vec{n}|+\beta_k+n_k+1}-1}\bar{\mathcal{M}}_{q,\vec{n}+\vec{e}_k}^{\vec{\beta}}f_{|\vec{n}|+1}(s)\\
&+b_{\vec{n},k}\bar{\mathcal{M}}_{q,\vec{n}}^{\vec{\beta}}f_{|\vec{n}|}(s)-q^{1/2}(1-\alpha q^{|\vec{n}|})\sum_{i=1}^{r}\prod_{j\neq i}^{r}\frac{x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)}\\
&\times\frac{x(n_i)}{1-\alpha q^{|\vec{n}|+\beta_i+n_i}}\frac{1}{1-\alpha q^{|\vec{n}|+\beta_i+n_i-1}}\prod_{l=1}^{r}\bar{\mathcal{M}}_{q,n_l-\delta_{l,i}}^{\beta_l}f_{|\vec{n}|-1}(s).
\end{aligned}
$$

Finally, multiplying from the left both sides of the previous expression by $\mathcal{G}_q^{\vec{n},\vec{\beta},\alpha}\frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i+s)}$ and using Rodrigues-type formula (2.3.14) we obtain (3.2.18), which completes the proof. □

3.3 Connection with the Multiple orthogonal polynomials

In this section we obtain the above theorem for the multiple orthogonal polynomials. Then, taking limit when $(q \to 1)$ in the above section, we get the following.

Let $\mathcal{F}_{n_i}$, where n_i is the i-th entry of the vector index $\vec{n}$, be a linear operator defined by

$$\mathcal{F}_{n_i} := g_i^{-1}(x) \bigtriangledown^{n_i} g_k(x), \tag{3.3.1}$$

where g_k is defined in the variable x and depends on the i-th component of the vector orthogonality measure $\vec{\mu}$. Moreover, if it also depends on the i-th component of $\vec{n}$, then the index $k = n_i$; otherwise $k = i$.

Lemma 3.3.1. *There holds the following relation*

$$\mathcal{F}_{n_i} x f(x) = n_i g_i^{-1}(x) \bigtriangledown^{n_i - 1} g_k(x) f(x) + (x - n_i) \mathcal{F}_{n_i} f(x). \tag{3.3.2}$$

Proof. It is enough to consider the relation,

$$\bigtriangledown^{n_i} x f(x) = n_i \bigtriangledown^{n_i - 1} f(x) + (x - n_i) \bigtriangledown^{n_i} f(x), \quad f(x) = g_k(x) f(x), \tag{3.3.3}$$

which is given in [12] to get (3.3.2) by multiplying on the left hand-side of $(3.3.3)$ by $g_i^{-1}(x)$. □

3.3.1 Multiple Hahn polynomials

Theorem 3.3.2. *The multiple Hahn orthogonal polynomials satisfy the following $(r+2)$-term recurrence relation*

$$\begin{aligned} x H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x) &= H_{\vec{n}+\vec{e}_k}^{\vec{\alpha},\alpha_0,N}(x) + b_{\vec{n}}^{\vec{\alpha},\alpha_0,N} H_{\vec{n}}^{\vec{\alpha},\alpha_0,N}(x) \\ &\quad + \sum_{1 \le i \le r} \prod_{j \ne i}^{r} \frac{n_i + \alpha_i - \alpha_j}{n_i - n_j + \alpha_i - \alpha_j} \prod_{1 \le j \le r} \frac{|\vec{n}| + \alpha_0 + \alpha_j}{|\vec{n}| + \alpha_0 + n_j + \alpha_j} \\ &\quad \times \frac{n_i (n_i + \alpha_i)(N + \alpha_0 + n_i + \alpha_i + 1)(N - |\vec{n}| + 1)(|\vec{n}| + \alpha_0)}{(|\vec{n}| + \alpha_0 + n_i + \alpha_i + 1)(|\vec{n}| + \alpha_0 + n_i + \alpha_i)(|\vec{n}| + \alpha_0 + n_i + \alpha_i - 1)} H_{\vec{n}-\vec{e}_i}^{\vec{\alpha},\alpha_0,N}(x), \end{aligned} \tag{3.3.4}$$

where

$$\begin{aligned} b_{\vec{n}}^{\vec{\alpha},\alpha_0,N} &= \prod_{1 \le i \le r} \frac{|\vec{n}| + \alpha_0 + \alpha_i + 1}{|\vec{n}| + \alpha_0 + n_i + \alpha_i + 1} \Bigg[\sum_{1 \le i \le r} \frac{n_i (N + \alpha_0 + n_i + \alpha_i + 1)}{|\vec{n}| + \alpha_0 + \alpha_i + 1} \\ &\quad + \left((N - |\vec{n}|) + \sum_{1 \le i \le r} (|\vec{n}| + \alpha_0 + n_i + \alpha_i + 1) \right) \prod_{1 \le i \le r} \frac{n_i}{|\vec{n}| + \alpha_0 + \alpha_i + 1} \\ &\quad + \frac{(n_k + \alpha_k + 1)(N - |\vec{n}|)}{|\vec{n}| + \alpha_0 + n_k + \alpha_k + 2} \Bigg] - \sum_{1 \le i \le r} \prod_{j \ne i}^{r} \frac{n_i + \alpha_i - \alpha_j}{n_i - n_j + \alpha_i - \alpha_j} \\ &\quad \times \frac{n_i (n_i + \alpha_i)(N + \alpha_0 + n_i + \alpha_i + 1)}{(|\vec{n}| + \alpha_0 + n_i + \alpha_i + 1)(|\vec{n}| + \alpha_0 + n_i + \alpha_i)}. \end{aligned}$$

Proof. First, let us consider equation

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha_k+x+1)}\,\nabla^{n_k+1}\,\Gamma(\alpha_k+n_k+1+x+1)\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-(|\vec{n}|+1)-x+1)}\\
=\;&\frac{1}{\Gamma(\alpha_k+x+1)}\,\nabla^{n_k}\left[\nabla\left(\Gamma(\alpha_k+n_k+1+x+1)\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-(|\vec{n}|+1)-x+1)}\right)\right]\\
=\;&\frac{1}{\Gamma(\alpha_k+x+1)}\,\nabla^{n_k}\left[(\Gamma(\alpha_k+n_k+x+1)\left((n_k+\alpha_k+1)\left(N-|\vec{n}|\right)\right.\right.\\
&\left.\left.-\left(|\vec{n}|+\alpha_0+n_k+\alpha_k+2\right)x\right)\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}\right)\right],
\end{aligned}
$$

which can be rewritten in terms of difference operators (1.2.21) as follows

$$
\begin{aligned}
&\mathcal{H}^{\alpha_k}_{n_k+1}\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-(|\vec{n}|+1)-x+1)}\\
&\qquad=(n_k+\alpha_k+1)\left(N-|\vec{n}|\right)\mathcal{H}^{\alpha_k}_{n_k}\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}\\
&\qquad\quad-\left(|\vec{n}|+\alpha_0+n_k+\alpha_k+2\right)\mathcal{H}^{\alpha_k}_{n_k}x\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}. \qquad (3.3.5)
\end{aligned}
$$

Since operators (1.2.21) commute, the multiplication of equation (3.3.5) from the left-hand side by the product $\left(\prod_{\substack{i=1\\ i\neq k}}^{r}\mathcal{H}^{\alpha_i}_{n_i}\right)$ yields

$$
\begin{aligned}
&\mathcal{H}^{\vec{\alpha}}_{\vec{n}}x\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}\\
&\qquad=-\frac{1}{|\vec{n}|+\alpha_0+n_k+\alpha_k+2}\mathcal{H}^{\vec{\alpha}}_{\vec{n}+\vec{e}_k}\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-(|\vec{n}|+1)-x+1)}\\
&\qquad\quad+\frac{(n_k+\alpha_k+1)(N-|\vec{n}|)}{|\vec{n}|+\alpha_0+n_k+\alpha_k+2}\mathcal{H}^{\vec{\alpha}}_{\vec{n}}\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}. \qquad (3.3.6)
\end{aligned}
$$

Second, let us recursively use Lemma 3.3.1 involving the product of r difference operators $\prod_{i=1}^{r}\mathcal{F}_{n_i}$ acting on the function $f_{|\vec{n}|}(s)=\frac{\Gamma(\alpha_0+N-x+1)}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)}$, which in this case is $\mathcal{H}^{\vec{\alpha}}_{\vec{n}}$ (see expression (1.2.21)). Thus, we have

$$\begin{aligned}
\mathcal{H}_{\vec{n}}^{\vec{\alpha}} x f_{|\vec{n}|}(x) = & \sum_{1\le i\le r} \prod_{j\ne i}^{r} \frac{n_i+\alpha_i-\alpha_j}{n_i-n_j+\alpha_i-\alpha_j} \\
& \times \frac{n_i\left(n_i+\alpha_i\right)\left(N+\alpha_0+n_i+\alpha_i+1\right)\left(|\vec{n}|+\alpha_0+n_j+\alpha_j+1\right)}{\prod\limits_{1\le\nu\le r}\left(|\vec{n}|+\alpha_0+\alpha_\nu+1\right)} \times \prod_{1\le l\le r} \mathcal{H}_{n_l-\delta_{l,i}}^{\alpha_l} f_{|\vec{n}|}(x) \\
& -\left(\sum_{1\le i\le r} \frac{n_i\left(N+\alpha_0+n_i+\alpha_i+1\right)}{|\vec{n}|+\alpha_0+\alpha_i+1} + \left((N-|\vec{n}|)+\sum_{1\le i\le r}\left(|\vec{n}|+\alpha_0+n_i+\alpha_i+1\right)\right)\right. \\
& \left.\times \prod_{1\le i\le r} \frac{n_i}{|\vec{n}|+\alpha_0+\alpha_i+1}\right) \mathcal{H}_{\vec{n}}^{\vec{\alpha}} f_{|\vec{n}|}(x) + \prod_{1\le i\le r} \frac{|\vec{n}|+\alpha_0+n_i+\alpha_i+1}{|\vec{n}|+\alpha_0+\alpha_i+1} x \mathcal{H}_{\vec{n}}^{\vec{\alpha}} f_{|\vec{n}|}(x). \quad (3.3.7)
\end{aligned}$$

Hence, by using expressions (3.3.6) and (3.3.7) one gets

$$\begin{aligned}
x\mathcal{H}_{\vec{n}}^{\vec{\alpha}} f_{|\vec{n}|}(x) = & -\prod_{1\le i\le r} \frac{|\vec{n}|+\alpha_0+\alpha_i+1}{|\vec{n}|+\alpha_0+n_i+\alpha_i+1} \frac{1}{|\vec{n}|+\alpha_0+n_k+\alpha_k+2} \mathcal{H}_{\vec{n}+\vec{e}_k}^{\vec{\alpha}} f_{|\vec{n}|+1}(x) \\
& + \prod_{1\le i\le r} \frac{|\vec{n}|+\alpha_0+\alpha_i+1}{|\vec{n}|+\alpha_0+n_i+\alpha_i+1} \left[\sum_{1\le i\le r} \frac{n_i\left(N+\alpha_0+n_i+\alpha_i+1\right)}{|\vec{n}|+\alpha_0+\alpha_i+1}\right. \\
& + \left((N-|\vec{n}|)+\sum_{1\le i\le r}\left(|\vec{n}|+\alpha_0+n_i+\alpha_i+1\right)\right) \prod_{1\le i\le r} \frac{n_i}{|\vec{n}|+\alpha_0+\alpha_i+1} \\
& \left. + \frac{(n_k+\alpha_k+1)(N-|\vec{n}|)}{|\vec{n}|+\alpha_0+n_k+\alpha_k+2}\right] \mathcal{H}_{\vec{n}}^{\vec{\alpha}} f_{|\vec{n}|}(x) - \sum_{1\le i\le r} \prod_{j\ne i}^{r} \frac{n_i+\alpha_i-\alpha_j}{n_i-n_j+\alpha_i-\alpha_j} \\
& \times \frac{n_i\left(n_i+\alpha_i\right)\left(N+\alpha_0+n_i+\alpha_i+1\right)}{\left(|\vec{n}|+\alpha_0+n_i+\alpha_i+1\right)} \prod_{1\le l\le r} \mathcal{H}_{n_l-\delta_{l,i}}^{\alpha_l} f_{|\vec{n}|}(x).
\end{aligned}$$

Observe that

$$\mathcal{H}_{n_l-\delta_{l,i}}^{\alpha_l} f_{|\vec{n}|}(x)\Big|_{l=i} = \frac{1}{|\vec{n}|+\alpha_0+n_i+\alpha_i} \mathcal{H}_{\vec{n}}^{\alpha_i} f_{|\vec{n}|}(x) + \frac{(N-|\vec{n}|+1)(|\vec{n}|+\alpha_0)}{|\vec{n}|+\alpha_0+n_i+\alpha_i} \mathcal{H}_{n_i}^{\alpha_i} f_{|\vec{n}|-1}(x).$$

Consequently

$$\begin{aligned}
x\mathcal{H}_{\vec{n}}^{\vec{\alpha}} f_{|\vec{n}|}(x) = & -\prod_{1\le i\le r} \frac{|\vec{n}|+\alpha_0+\alpha_i+1}{|\vec{n}|+\alpha_0+n_i+\alpha_i+1} \frac{1}{|\vec{n}|+\alpha_0+n_k+\alpha_k+2} \mathcal{H}_{\vec{n}+\vec{e}_k}^{\vec{\alpha}} f_{|\vec{n}|+1}(x) \\
& + b_{\vec{n}}^{\vec{\alpha},\alpha_0,N} \mathcal{H}_{\vec{n}}^{\vec{\alpha}} f_{|\vec{n}|}(x) - \sum_{1\le i\le r} \prod_{j\ne i}^{r} \frac{n_i+\alpha_i-\alpha_j}{n_i-n_j+\alpha_i-\alpha_j} \\
& \times \frac{n_i\left(n_i+\alpha_i\right)\left(N+\alpha_0+n_i+\alpha_i+1\right)\left(N-|\vec{n}|+1\right)\left(|\vec{n}|+\alpha_0\right)}{\left(|\vec{n}|+\alpha_0+n_i+\alpha_i+1\right)\left(|\vec{n}|+\alpha_0+n_i+\alpha_i\right)} \prod_{1\le l\le r} \mathcal{H}_{n_l-\delta_{l,i}}^{\alpha_l} f_{|\vec{n}|-1}(x).
\end{aligned}$$

Finally, multiplying from the left both sides of the previous expression by $\frac{(-1)^{|\vec{n}|}}{\prod_{i=1}^{r}(|\vec{n}|+\alpha_i+\alpha_0+1)_{n_i}}\frac{\Gamma(x+1)\Gamma(N-x+1)}{\Gamma(\alpha_0+N-x+1)}$ and using Rodrigues-type formula (1.2.20) we obtain (3.2.18), which completes the proof. □

Notice that the approach used to obtain (3.3.4) is new in the literature and (1.2.6), (1.2.10), (1.2.14), and (1.2.18) we can be obtained in a similar way.

In addition, Lemma 3.3.1 allows to obtain recurrence relations for polynomial families defined by means of Rodrigues-type formulas without verifying orthogonality conditions (1.3.1). These families will be called, in the sequel, unusual cases.

3.3.2 Unusual case

Suppose that the following Rodrigues-type formula holds

$$Q_{\vec{n}}^{\vec{\beta},\alpha} = \left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\frac{\Gamma(x+1)}{\alpha^x}\mathcal{Q}_{\vec{n}}^{\vec{\beta}}\frac{\alpha^x}{\Gamma(x+1)}, \tag{3.3.8}$$

where

$$\mathcal{Q}_{\vec{n}}^{\vec{\beta}} = \prod_{i=1}^{r}\mathcal{Q}_{n_i}^{\beta_i}, \quad \mathcal{Q}_{n_i}^{\beta_i} = \prod_{i=1}^{r}\left(\frac{1}{\Gamma(\beta_i+x+1)}\nabla^{n_i}\,\Gamma(\beta_i+n_i+x+1)\right), \tag{3.3.9}$$

with $0<\alpha<1$, $\beta_i>0$ and $\beta_i\neq\beta_j$ whenever $i\neq j$.

Theorem 3.3.3. *The expression* (3.3.8) *satisfies the following* $(r+2)$*-term recurrence relation*

$$\begin{aligned} xQ_{\vec{n}}^{\vec{\beta},\alpha}(x) &= Q_{\vec{n}+\vec{e}_k}^{\vec{\beta},\alpha}(x) + \left[(n_k+\beta_k+1)\frac{\alpha}{1-\alpha}+\sum_{i=1}^{r}\frac{n_i}{1-\alpha}\right]Q_{\vec{n}}^{\vec{\beta},\alpha}(x) \\ &\quad + \sum_{1\le i\le r} n_i\,(n_i+\beta_i)\frac{\alpha}{(1-\alpha)^2}\prod_{j\neq i}^{r}\frac{n_j+\beta_j-\beta_i}{n_j-n_i+\beta_j-\beta_i}Q_{\vec{n}-\vec{e}_i}^{\vec{\beta},\alpha}(x). \end{aligned} \tag{3.3.10}$$

Proof. First, let us consider

$$\begin{aligned} &\frac{1}{\Gamma(\beta_k+x+1)}\nabla^{n_k+1}\,\Gamma(\beta_k+n_k+1+x+1)\frac{\alpha^x}{\Gamma(x+1)} \\ =\;& \frac{1}{\Gamma(\beta_k+x+1)}\nabla^{n_k}\left[\nabla\left(\Gamma(\beta_k+n_k+1+x+1)\frac{\alpha^x}{\Gamma(x+1)}\right)\right] \\ =\;& \frac{1}{\Gamma(\beta_k+x+1)}\nabla^{n_k}\left[\left(\Gamma(\beta_k+n_k+x+1)\left((n_k+\beta_k+1)-\frac{(1-\alpha)\,x}{\alpha}\right)\frac{\alpha^x}{\Gamma(x+1)}\right)\right], \end{aligned}$$

which can be rewritten in terms of difference operators (3.3.9) as follows

$$\mathcal{Q}_{n_k+1}^{\beta_k}\frac{\alpha^x}{\Gamma(x+1)} = (n_k+\beta_k+1)\,\mathcal{Q}_{n_k}^{\beta_k}\frac{\alpha^x}{\Gamma(x+1)} - \left(\frac{1-\alpha}{\alpha}\right)\mathcal{Q}_{n_k}^{\beta_k}x\frac{\alpha^x}{\Gamma(x+1)}. \tag{3.3.11}$$

Since operators (3.3.9) commute, the multiplication of equation (3.3.11) from the left-hand side by the product $\left(\prod_{\substack{i=1\\i\neq k}}^{r} \mathcal{Q}_{n_i}^{\beta_i}\right)$ yields

$$\mathcal{Q}_{\vec{n}}^{\vec{\beta}} x \frac{\alpha^x}{\Gamma(x+1)} = \frac{\alpha}{\alpha-1} \mathcal{Q}_{\vec{n}+\vec{e}_k}^{\vec{\beta}} \frac{\alpha^x}{\Gamma(x+1)} + \frac{(n_k+\alpha_k+1)\alpha}{1-\alpha} \mathcal{Q}_{\vec{n}}^{\vec{\beta}} \frac{\alpha^x}{\Gamma(x+1)}. \tag{3.3.12}$$

Second, let us recursively use Lemma 3.3.1 involving the product of r difference operators $\prod_{i=1}^{r} \mathcal{F}_{n_i}$ acting on the function $f(s) = \frac{\alpha^x}{\Gamma(x+1)}$, which in this case is $\mathcal{Q}_{\vec{n}}^{\vec{\beta}}$ (see expression (3.3.9)). Thus, we have

$$\mathcal{Q}_{\vec{n}}^{\vec{\beta}} x f(x) = \sum_{1\leq i\leq r} \prod_{j\neq i}^{r} \frac{n_i+\beta_i-\beta_j}{n_i-n_j+\beta_i-\beta_j} \frac{n_i\,(n_i+\beta_i)}{1-\alpha} \prod_{1\leq l\leq r} \mathcal{Q}_{n_l-\delta_{l,i}}^{\beta_l} f(x) \\ - \sum_{1\leq i\leq r} \frac{n_i}{1-\alpha} \mathcal{Q}_{\vec{n}}^{\vec{\beta}} f(x) + x \mathcal{Q}_{\vec{n}}^{\vec{\beta}} f(x). \tag{3.3.13}$$

Hence, by using expressions (3.3.12), (3.3.13) one gets

$$x \mathcal{Q}_{\vec{n}}^{\vec{\beta}} f(x) = \frac{\alpha}{\alpha-1} \mathcal{Q}_{\vec{n}+\vec{e}_k}^{\vec{\beta}} f(x) \\ + \left[\sum_{1\leq i\leq r} \frac{n_i}{1-\alpha} + \frac{(n_k+\beta_k+1)\alpha}{1-\alpha}\right] \mathcal{Q}_{\vec{n}}^{\vec{\beta}} f(x) \\ - \sum_{1\leq i\leq r} \prod_{j\neq i}^{r} \frac{n_i+\beta_i-\beta_j}{n_i-n_j+\beta_i-\beta_j} \frac{n_i\,(n_i+\beta_i)}{1-\alpha} \prod_{1\leq l\leq r} \mathcal{Q}_{n_l-\delta_{l,i}}^{\alpha_l} f(x).$$

Finally, multiplying from the left both sides of the previous expression by $\left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|} \frac{\Gamma(x+1)}{\alpha^x}$ and using Rodrigues-type formula (3.3.8) we obtain (3.3.10), which completes the proof. □

Chapter 4

Limit relations

We would like to recall that the selected discretization; that is, the q-lattice $x(s) = (q^s - 1)/(q - 1)$ allows easily to transit from the non-uniform distribution of mass points $x(s)$, $s = 0, 1, \dots,$ to the uniform distribution $x(s) = s$, when q approaches 1. Under this limiting process one expects that the corresponding q-algebraic relations transform into the discrete relations; i.e. Rodrigues-type formulas as well as recurrence relations, among others, will be transformed in their discrete counterparts. Thus, in this chapter we focus our attention on some limit relations between q-multiple orthogonal polynomials and discrete multiple orthogonal polynomials. Indeed, we will start by analyzing the Rodrigues-type formulas, which we will use as a main ingredient for addressing the limit relations involving the difference equations and recurrence relations, respectively.

4.1 q-Rodrigues type formula

In this section we study the limit relations for the Rodrigues-type formula.

4.1.1 q-Charlier multiple orthogonal polynomials

Proposition 4.1.1. *The following limiting relation is valid:*

$$\lim_{q\to 1} C_{q,\vec{n}}^{\vec{\alpha}}(s) = C_{\vec{n}}^{\vec{\alpha}}(s), \tag{4.1.1}$$

where $C_{\vec{n}}^{\vec{\alpha}}(s)$ and $C_{q,\vec{n}}^{\vec{\alpha}}(s)$ are defined in (1.2.4) *and* (2.3.5)*, respectively.*

Proof. From (2.3.5) we have

$$C_{q,\vec{n}}^{\vec{\alpha}}(s) = (-1)^{|\vec{n}|} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^{r} \alpha_i^{n_i}\right) \left(\prod_{i=1}^{r} q^{n_i \sum_{j=i}^{r} n_j}\right) \Gamma_q(s+1) \prod_{i=1}^{r} (\alpha_i)^{-s} \nabla^{n_i} (\alpha_i q^{n_i})^s \frac{1}{\Gamma_q(s+1)}.$$

Thus, by (1.3.4), (1.3.5), and $\triangledown x_1(s) \stackrel{\text{def}}{=} \triangledown x(s+1/2) = \triangle x(s-1/2) = q^{s-1/2}$ we get

$$C^{\vec{\alpha}}_{q,\vec{n}}(s) = (-1)^{|\vec{n}|} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^{r} \alpha_i^{n_i}\right) \left(\prod_{i=1}^{r} q^{n_i \sum_{j=i}^{r} n_j}\right) \Gamma_q(s+1)$$
$$\times \prod_{i=1}^{r} (\alpha_i)^{-s} \triangledown^{n_i} (\alpha_i q^{n_i})^s \frac{q^{n_i(s-1/2)}}{\Gamma_q(s+1)}. \quad (4.1.2)$$

Now, taking into account that

$$\Gamma_q(s+1) = x(s)\Gamma_q(s) = \cdots = x(s)! \quad \text{and} \quad x(s) = \frac{q^s-1}{q-1}. \quad (4.1.3)$$

Applying limit in (4.1.2) when q tends to 1 we get

$$\lim_{q\to 1} C^{\vec{\alpha}}_{q,\vec{n}}(s) = (-1)^{|\vec{n}|} \left(\prod_{i=1}^{r} \alpha_i^{n_i}\right) \Gamma(s+1) \prod_{i=1}^{r} (\alpha_i)^{-s} \triangledown^{n_i} (\alpha_i)^s \frac{1}{\Gamma(s+1)},$$

which proves the above expression (4.1.1). □

4.1.2 q-Meixner multiple orthogonal polynomials of the first kind

Proposition 4.1.2. *The following limiting relation is valid:*

$$\lim_{q\to 1} M^{\vec{\alpha},\beta}_{q,\vec{n}}(s) = M^{\vec{\alpha},\beta}_{\vec{n}}(s), \quad (4.1.4)$$

where $M^{\vec{\alpha},\beta}_{\vec{n}}(s)$ and $M^{\vec{\alpha},\beta}_{q,\vec{n}}(s)$ are defined in (1.2.8) *and* (2.3.9)*, respectively.*

Proof. From (2.3.9) we have

$$M^{\vec{\alpha},\beta}_{q,\vec{n}}(s) = (-1)^{|\vec{n}|} [-\beta]_q^{(|\vec{n}|)} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^{r} \frac{\alpha_i^{n_i} \prod_{j=1}^{n_i} q^{|\vec{n}|_i+\beta+j-1}}{\prod_{j=1}^{n_i} (\alpha_i q^{|\vec{n}|+\beta+j-1} - 1)}\right) \left(\prod_{i=1}^{r} q^{n_i \sum_{j=i}^{r} n_j}\right)$$
$$\times \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)} \prod_{i=1}^{r} (\alpha_i)^{-s} \nabla^{n_i} (\alpha_i q^{n_i})^s \left(\frac{\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}\right).$$

Thus, by (1.3.4) and (1.3.5) we get

$$M^{\vec{\alpha},\beta}_{q,\vec{n}}(s) = (-1)^{|\vec{n}|} [-\beta]_q^{(|\vec{n}|)} q^{-\frac{|\vec{n}|}{2}} \left(\prod_{i=1}^{r} \frac{\alpha_i^{n_i} \prod_{j=1}^{n_i} q^{|\vec{n}|_i+\beta+j-1}}{\prod_{j=1}^{n_i} (\alpha_i q^{|\vec{n}|+\beta+j-1} - 1)}\right) \left(\prod_{i=1}^{r} q^{n_i \sum_{j=i}^{r} n_j}\right)$$
$$\times \frac{\Gamma_q(\beta)\Gamma_q(s+1)}{\Gamma_q(\beta+s)} \prod_{i=1}^{r} (\alpha_i)^{-s} \triangledown^{n_i} (\alpha_i q^{n_i})^s \left(\frac{q^{n_i(s-1/2)}\Gamma_q(\beta+|\vec{n}|+s)}{\Gamma_q(\beta+|\vec{n}|)\Gamma_q(s+1)}\right).$$

Now, by (4.1.3) and applying limit in the above expression when q goes to 1 we get

$$\lim_{q\to 1} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = (\beta)_{|\vec{n}|}\left[\prod_{i=1}^{r}\left(\frac{\alpha_i}{\alpha_i-1}\right)^{n_i}\right]\frac{\Gamma(\beta)\Gamma(x+1)}{\Gamma(\beta+x)}\prod_{i=1}^{r}(\alpha_i)^{-s}\bigtriangledown^{n_i}(\alpha_i)^{s}\left(\frac{\Gamma(\beta+|\vec{n}|+x)}{\Gamma(\beta+|\vec{n}|)\Gamma(x+1)}\right),$$

which proves the above expression (4.1.4). □

4.1.3 q-Meixner multiple orthogonal polynomials of the second kind

Proposition 4.1.3. *The following limiting relation is valid:*

$$\lim_{q\to 1} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = M_{\vec{n}}^{\vec{\beta},\alpha}(s), \tag{4.1.5}$$

where $M_{\vec{n}}^{\vec{\beta},\alpha}(s)$ and $M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$ are defined in (1.2.12) *and* (2.3.14)*, respectively.*

Proof. From (2.3.14) we have

$$M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = (-1)^{|\vec{n}|}\left(\alpha q^{|\vec{n}|}\right)^{|\vec{n}|} q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\frac{\prod_{j=1}^{n_i} q^{\beta_i+j-1}}{\prod_{j=1}^{n_i}\left(\alpha q^{|\vec{n}|+\beta_i+j-1}-1\right)}\right)\left(\prod_{i=1}^{r}[-\beta_i]_q^{(n_i)}\right)$$
$$\times\frac{\Gamma_q(s+1)}{\alpha^s}\prod_{i=1}^{r}\frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i+s)}\nabla^{n_i}\frac{\Gamma_q(\beta_i+n_i+s)}{\Gamma_q(\beta_i+n_i)}\left(\frac{\left(\alpha q^{|\vec{n}|}\right)^s}{\Gamma_q(s+1)}\right).$$

Thus, by (1.3.4) and (1.3.5) we get

$$M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = (-1)^{|\vec{n}|}\left(\alpha q^{|\vec{n}|}\right)^{|\vec{n}|} q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\frac{\prod_{j=1}^{n_i} q^{\beta_i+j-1}}{\prod_{j=1}^{n_i}\left(\alpha q^{|\vec{n}|+\beta_i+j-1}-1\right)}\right)\left(\prod_{i=1}^{r}[-\beta_i]_q^{(n_i)}\right)$$
$$\times\frac{\Gamma_q(s+1)}{\alpha^s}\prod_{i=1}^{r}\frac{\Gamma_q(\beta_i)}{\Gamma_q(\beta_i+s)}\bigtriangledown^{n_i}\frac{\Gamma_q(\beta_i+n_i+s)}{\Gamma_q(\beta_i+n_i)}\left(\frac{q^{n_i(s-1/2)}\left(\alpha q^{|\vec{n}|}\right)^s}{\Gamma_q(s+1)}\right).$$

Now, by (4.1.3) and applying limit in the above expression when q approaches to 1 we get

$$\lim_{q\to 1} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = \left(\frac{\alpha}{\alpha-1}\right)^{|\vec{n}|}\left[\prod_{i=1}^{r}(\beta_i)_{n_i}\right]\frac{\Gamma(x+1)}{\alpha^x}\prod_{i=1}^{r}\frac{\Gamma(\beta_i)}{\Gamma(\beta_i+x)}\bigtriangledown^{n_i}\frac{\Gamma(\beta_i+n_i+x)}{\Gamma(\beta_i+n_i)}\left(\frac{\alpha^x}{\Gamma(x+1)}\right),$$

which completes the proof of relation (4.1.5). □

4.1.4 q-Kravchuk multiple orthogonal polynomials

Proposition 4.1.4. *The following limiting relation is valid:*

$$\lim_{q\to 1} K_{q,\vec{n}}^{\vec{p},N}(s) = K_{\vec{n}}^{\vec{p},N}(s), \tag{4.1.6}$$

where $K_{\vec{n}}^{\vec{p},N}(s)$ *and* $K_{q,\vec{n}}^{\vec{p},N}(s)$ *are defined in* (1.2.16) *and* (2.3.18), *respectively.*

Proof. From (2.3.18) we have

$$K_{q,\vec{n}}^{\vec{p},N}(s) = (-1)^{|\vec{n}|}[N]_q^{(|\vec{n}|)}q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\frac{p_i^{n_i}}{\prod\limits_{j=1}^{n_i} q^{-j}\left[p_i\left(q^{|\vec{n}|-|\vec{n}|_i-j-1}-1\right)+1\right]}\right)q^{-2|\vec{n}|}\left(\prod_{i=1}^{r-1} q^{n_i \sum\limits_{j=i+1}^{r} n_j}\right)$$
$$\times \frac{\Gamma_q(N-s+1)\Gamma_q(s+1)}{q^{\binom{s}{2}}[N]_q!}\prod_{i=1}^{r}\left(\frac{p_i}{1-p_i}\right)^{-s}\nabla^{n_i}\left(\frac{p_i q^{2n_i}}{1-p_i}\right)^{s}\left(\frac{q^{\binom{s}{2}}[N-|\vec{n}|]_q!}{\Gamma_q(N-|\vec{n}|-s+1)\Gamma_q(s+1)}\right).$$

Thus, by (1.3.4) and (1.3.5)

$$K_{q,\vec{n}}^{\vec{p},N}(s) = (-1)^{|\vec{n}|}[N]_q^{(|\vec{n}|)}q^{-\frac{|\vec{n}|}{2}}\left(\prod_{i=1}^{r}\frac{p_i^{n_i}}{\prod\limits_{j=1}^{n_i} q^{-j}\left[p_i\left(q^{|\vec{n}|-|\vec{n}|_i-j-1}-1\right)+1\right]}\right)q^{-2|\vec{n}|}\left(\prod_{i=1}^{r-1} q^{n_i \sum\limits_{j=i+1}^{r} n_j}\right)$$
$$\times \frac{\Gamma_q(N-s+1)\Gamma_q(s+1)}{q^{\binom{s}{2}}[N]_q!}\prod_{i=1}^{r}\left(\frac{p_i}{1-p_i}\right)^{-s}\triangledown^{n_i}\left(\frac{p_i q^{2n_i}}{1-p_i}\right)^{s}\left(\frac{q^{n_i(s-1/2)}q^{\binom{s}{2}}[N-|\vec{n}|]_q!}{\Gamma_q(N-|\vec{n}|-s+1)\Gamma_q(s+1)}\right).$$

Now, by (4.1.3) and applying limit in the above expression when q goes to 1 we get

$$\lim_{q\to 1} K_{q,\vec{n}}^{\vec{p},N}(s) = (-N)_{|\vec{n}|}\left(\prod_{i=1}^{r} p_i^{n_i}\right)\frac{\Gamma(x+1)\Gamma(N-x+1)}{N!}$$
$$\times \prod_{i=1}^{r}\left(\frac{1-p_i}{p_i}\right)^{x}\triangledown^{n_i}\left(\frac{p_i}{1-p_i}\right)^{x}\frac{(N-|\vec{n}|)!}{\Gamma(x+1)\Gamma(N-|\vec{n}|-x+1)},$$

which proves the above expression (4.1.6). □

4.2 q-Difference equation

The previous propositions based on the limit relation of the Rodrigues-type formula of these polynomials, also allows to obtain limit relation between the difference equations. In this section we show limit relations for difference equation.

4.2.1 q-Charlier multiple orthogonal polynomials

Proposition 4.2.1. *The following limiting relation is valid:*

$$\lim_{q\to 1}\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)=\prod_{i=1}^{r}\mathcal{L}^{\alpha_i}\triangle C_{\vec{n}}^{\vec{\alpha}}(s), \tag{4.2.1}$$

where $\mathcal{L}^{\alpha_i}$ and $\mathcal{D}_q^{\alpha_i}$ are defined in (1.2.3) *and* (2.3.4)*, respectively.*

Proof. By Theorem 3.1.3 we have

$$\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)=-\sum_{i=1}^{r}q^{|\vec{n}|-n_i+1}[n_i]_q^{(1)}\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j}C_{q,\vec{n}}^{\vec{\alpha}}(s).$$

From (2.3.4) and (3.1.8) we get

$$\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)=\sum_{i=1}^{r}q^{|\vec{n}|-n_i+3/2}\frac{q^{n_i}-1}{q-1}\prod_{\substack{j=1\\ j\neq i}}^{r}C_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha}}(s).$$

Applying limit in the above expression when q goes to 1 and by Proposition 4.1.1 we get

$$\lim_{q\to 1}\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i}\Delta C_{q,\vec{n}}^{\vec{\alpha}}(s)=\sum_{i=1}^{r}n_i\prod_{\substack{j=1\\ j\neq i}}^{r}C_{\vec{n}+\vec{e}_j}^{\vec{\alpha}}(s).$$

Finally, from (1.2.3) and (1.2.5), (4.2.1) holds. □

4.2.2 q-Meixner multiple orthogonal polynomials of the first kind

Proposition 4.2.2. *The following limiting relation is valid:*

$$\lim_{q\to 1}\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1+i-r}\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)=\prod_{i=1}^{r}\mathcal{L}^{\alpha_i,\beta+1+i-r}\triangle M_{\vec{n}}^{\vec{\alpha},\beta}(s), \tag{4.2.2}$$

where $\mathcal{L}^{\alpha_i,\beta}$ and $\mathcal{D}_q^{\alpha_i,\beta}$ are defined in (1.2.7) *and* (2.3.8)*, respectively.*

Proof. By Theorem 3.1.6 we have

$$\prod_{i=1}^{r}\mathcal{D}_q^{q\alpha_i,\beta+1+i-r}\Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)=-\sum_{i=1}^{r}q^{|\vec{n}|-n_i+1}\frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}}[n_i]_q^{(1)}\prod_{\substack{j=1\\ j\neq i}}^{r}\mathcal{D}_q^{q\alpha_j,\beta+1+j-r}M_{q,\vec{n}}^{\vec{\alpha},\beta}(s).$$

From (2.3.8) and (3.1.8) we get

$$\prod_{i=1}^{r} \mathcal{D}_q^{q\alpha_i,\beta+1+i-r} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \sum_{i=1}^{r} q^{|\vec{n}|-n_i+3/2} \frac{1-\alpha_i q^{n_i+\beta}}{1-\alpha_i q^{|\vec{n}|+\beta}} \frac{q^{n_i}-1}{q-1} \prod_{\substack{j=1\\ j\neq i}}^{r} M_{q,\vec{n}+\vec{e}_j}^{\vec{\alpha},\beta+j-r}(s).$$

Applying limit in the above expression when q goes to 1 and by Proposition 4.1.2 we get

$$\lim_{q\to 1} \prod_{i=1}^{r} \mathcal{D}_q^{q\alpha_i,\beta+1+i-r} \Delta M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = \sum_{i=1}^{r} n_i \prod_{\substack{j=1\\ j\neq i}}^{r} M_{\vec{n}+\vec{e}_j}^{\vec{\alpha},\beta+j-r}(s).$$

Finally, from (1.2.7) and (1.2.9), (4.2.2) holds. □

4.2.3 q-Meixner multiple orthogonal polynomials of the second kind

Proposition 4.2.3. *The following limiting relation is valid:*

$$\lim_{q\to 1} \prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = \prod_{i=1}^{r} \mathcal{L}^{\beta_i+1,\alpha} \triangle M_{\vec{n}}^{\vec{\beta},\alpha}(s), \tag{4.2.3}$$

where $\mathcal{L}^{\beta_i,\alpha}$ *and* $\mathcal{D}_q^{\beta_i,\alpha}$ *are defined in* (1.2.11) *and* (2.3.12)*, respectively.*

Proof. By Theorem 3.1.11 we have

$$\prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = -\sum_{i=1}^{r} q^{1/2}\xi_i \prod_{\substack{j=1\\ j\neq i}}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_j+1,q\alpha} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s),$$

where ξ_i's are the constants in (3.1.25).
From (2.3.12) we get

$$\prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = \sum_{i=1}^{r} q\xi_i \prod_{\substack{j=1\\ j\neq i}}^{r} M_{q,\vec{n}+\vec{e}_j}^{\vec{\beta},\alpha}(s).$$

Applying limit in the above expression when q goes to 1 and by Proposition 4.1.3 we get

$$\lim_{q\to 1} \prod_{i=1}^{r} \mathcal{D}_{q,\vec{n}}^{\beta_i+1,q\alpha} \Delta M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = \sum_{i=1}^{r} d_i \prod_{\substack{j=1\\ j\neq i}}^{r} M_{\vec{n}+\vec{e}_j}^{\vec{\beta},\alpha}(s),$$

where d_i's are the constants in (1.2.13).
Finally, from (1.2.11) and (1.2.13), (4.2.3) holds. □

4.2.4 q-Kravchuk multiple orthogonal polynomials

Proposition 4.2.4. *The following limiting relation is valid:*

$$\lim_{q\to 1}\prod_{i=1}^{r}\mathcal{D}_q^{p_i,N+r-i-1,\beta_i/q^2}\Delta K_{q,\vec{n}}^{\vec{p},N}(s)=\prod_{i=1}^{r}\mathcal{L}^{p_i,N+r-i-1}\triangle K_{\vec{n}}^{\vec{p},N}(s), \tag{4.2.4}$$

where $\mathcal{L}^{p_i,N}$ and $\mathcal{D}_q^{p_i,N}$ are defined in (1.2.15) *and* (2.3.17), *respectively.*

Proof. By Theorem 3.1.14 we have

$$\prod_{i=1}^{r}\mathcal{D}_q^{p_i,N-1+r-i,\beta_i/q^2}\Delta K_{q,\vec{n}}^{\vec{p},N}(s)$$
$$=-\sum_{i=1}^{r}q^{|\vec{n}|-n_i+1}\frac{[p_i\left(q^{n_i}-1\right)+1]}{[p_i\left(q^{|\vec{n}|}-1\right)+1]}[n_i]_q^{(1)}\prod_{\substack{j=1\\j\neq i}}^{r}\mathcal{D}_q^{p_j,N-1+r-j,\beta_j/q^2}K_{q,\vec{n}}^{\vec{p},N}(s).$$

From (2.3.17) and (3.1.8) we get

$$\prod_{i=1}^{r}\mathcal{D}_q^{p_i,N-1+r-i,\beta_i/q^2}\Delta K_{q,\vec{n}}^{\vec{p},N}(s)=\sum_{i=1}^{r}q^{|\vec{n}|-n_i+3/2}\frac{[p_i\left(q^{n_i}-1\right)+1]}{[p_i\left(q^{|\vec{n}|}-1\right)+1]}\frac{q^{n_i}-1}{q-1}\prod_{\substack{j=1\\j\neq i}}^{r}K_{q,\vec{n}+\vec{e}_j}^{\vec{p},N+r-j}(s).$$

Applying limit in the above expression when q approaches to 1 and by Proposition 4.1.4 we get

$$\lim_{q\to 1}\prod_{i=1}^{r}\mathcal{D}_q^{p_i,N-1+r-i,\beta_i/q^2}\Delta K_{q,\vec{n}}^{\vec{p},N}(s)=\sum_{i=1}^{r}n_i\prod_{\substack{j=1\\j\neq i}}^{r}K_{\vec{n}+\vec{e}_j}^{\vec{p},N+r-j}(s).$$

Finally, from (1.2.15) and (1.2.17), (4.2.4) holds. □

4.3 q-Recurrence relation

Finally, this section shows the limit relations of the recurrence relations of these polynomials.

4.3.1 q-Charlier multiple orthogonal polynomials

Proposition 4.3.1. *The following limiting relation is valid:*

$$\lim_{q\to 1}x(s)C_{q,\vec{n}}^{\vec{\alpha}}(s)=C_{\vec{n}+\vec{e}_k}^{\vec{\alpha}}(s)+(|\vec{n}|+\alpha_k)C_{\vec{n}}^{\vec{\alpha}}(s)+\sum_{i=1}^{r}\alpha_i n_i C_{\vec{n}-\vec{e}_i}^{\vec{\alpha}}(s), \tag{4.3.1}$$

where the right-hand side of (4.3.1) *and the product term $x(s)C_{q,\vec{n}}^{\vec{\alpha}}(s)$ are defined in* (1.2.6) *and* (3.2.5), *respectively.*

Proof. From Theorem 3.2.2 we have

$$x(s)C_{q,\vec{n}}^{\vec{\alpha}}(s) = C_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha}}(s) + \left(\sum_{i=1}^{r} q^{|\vec{n}|_i} x(n_i) A_{\vec{n},i} + \alpha_k q^{|\vec{n}|+n_k+1}\right) C_{q,\vec{n}}^{\vec{\alpha}}(s) + q^{1/2}\sum_{i=1}^{r} x(n_i)\alpha_i q^{|\vec{n}|+n_i-1} B_{\vec{n},i} C_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha}}(s),$$

where $A_{\vec{n},i}$ and $B_{\vec{n},i}$ are defined in (3.2.5).

Notice that, applying limit in both side of the above expression when q goes to 1 and by (3.1.8) we have

$$\lim_{q\to 1}\left(\sum_{i=1}^{r} q^{|\vec{n}|_i}\frac{q^{n_i}-1}{q-1} A_{\vec{n},i} + \alpha_k q^{|\vec{n}|+n_k+1}\right) = |\vec{n}| + \alpha_k,$$

$$\lim_{q\to 1}\left(q^{1/2}\frac{q^{n_i}-1}{q-1}\alpha_i q^{|\vec{n}|+n_i-1} B_{\vec{n},i}\right) = \alpha_i n_i,$$

and by Proposition 4.1.1 we get

$$\lim_{q\to 1} x(s)C_{q,\vec{n}}^{\vec{\alpha}}(s) = C_{\vec{n}+\vec{e_k}}^{\vec{\alpha}}(s) + (|\vec{n}| + \alpha_k)C_{\vec{n}}^{\vec{\alpha}}(s) + \sum_{i=1}^{r}\alpha_i n_i C_{\vec{n}-\vec{e}_i}^{\vec{\alpha}}(s),$$

which completes the proof of the above expression (4.3.1). □

4.3.2 q-Meixner multiple orthogonal polynomials of the first kind

Proposition 4.3.2. *The following limiting relation is valid:*

$$\lim_{q\to 1} x(s)M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = M_{\vec{n}+\vec{e_k}}^{\vec{\alpha},\beta}(s) + \left[(\beta+|\vec{n}|)\left(\frac{\alpha_k}{1-\alpha_k}\right) + \sum_{i=1}^{r}\frac{n_i}{1-\alpha_i}\right] M_{\vec{n}}^{\vec{\alpha}\beta}(s) + \sum_{i=1}^{r}\frac{\alpha_i n_i\left(\beta+|\vec{n}|-1\right)}{\left(\alpha_i-1\right)^2} M_{\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta}(s), \quad (4.3.2)$$

where the right-hand side of (4.3.2) *and the product term* $x(s)M_{q,\vec{n}}^{\vec{\alpha},\beta}(s)$ *are defined in* (1.2.10) *and* (3.2.14)*, respectively.*

Proof. From Theorem 3.2.3 we have

$$x(s)M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = M_{q,\vec{n}+\vec{e}_k}^{\vec{\alpha},\beta}(s) + b_{\vec{n},k} M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) + \sum_{i=1}^{r}\frac{x(n_i)\alpha_i q^{|\vec{n}|+n_i-1} x(\beta+|\vec{n}|-1)}{\left(\alpha_i q^{|\vec{n}|+\beta+n_i-1}-1\right)\left(\alpha_i q^{|\vec{n}|+\beta+n_i-2}-1\right)} B_{\vec{n},i} M_{q,\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta}(s),$$

where $b_{\vec{n},k}$ and $B_{\vec{n},i}$ are defined in (3.2.14).

Notice that, applying limit in both side of the above expression when q goes to 1 and by (3.1.8) we have

$$\lim_{q\to 1} b_{\vec{n},k} = (\beta + |\vec{n}|)\left(\frac{\alpha_k}{1-\alpha_k}\right) + \sum_{i=1}^{r} \frac{n_i}{1-\alpha_i},$$
$$\lim_{q\to 1}\left(\frac{(q^{n_i}-1)(q^{\beta+|\vec{n}|-1}-1)\alpha_i q^{|\vec{n}|+n_i-1}}{(q-1)^2\left(\alpha_i q^{|\vec{n}|+\beta+n_i-1}-1\right)\left(\alpha_i q^{|\vec{n}|+\beta+n_i-2}-1\right)} B_{\vec{n},i}\right) = \frac{\alpha_i n_i\left(\beta+|\vec{n}|-1\right)}{\left(\alpha_i-1\right)^2},$$

and by Proposition 4.1.2 we get

$$\lim_{q\to 1} x(s) M_{q,\vec{n}}^{\vec{\alpha},\beta}(s) = M_{\vec{n}+\vec{e}_k}^{\vec{\alpha},\beta}(s) + \left[(\beta+|\vec{n}|)\left(\frac{\alpha_k}{1-\alpha_k}\right) + \sum_{i=1}^{r}\frac{n_i}{1-\alpha_i}\right] M_{\vec{n}}^{\vec{\alpha}\beta}(s)$$
$$+ \sum_{i=1}^{r} \frac{\alpha_i n_i\left(\beta+|\vec{n}|-1\right)}{\left(\alpha_i-1\right)^2} M_{\vec{n}-\vec{e}_i}^{\vec{\alpha},\beta}(s),$$

which proves the above expression (4.3.2). □

4.3.3 q-Meixner multiple orthogonal polynomials of the second kind

Proposition 4.3.3. *The following limiting relation is valid:*

$$\lim_{q\to 1} x(s) M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = M_{\vec{n}+\vec{e}_k}^{\vec{\beta},\alpha}(s) + \left[(n_k+\beta_k)\left(\frac{\alpha}{1-\alpha}\right) + \frac{|\vec{n}|}{1-\alpha}\right] M_{\vec{n}}^{\vec{\beta},\alpha}(s)$$
$$+ \alpha \sum_{i=1}^{r} \frac{n_i\left(\beta_i+n_i-1\right)}{\left(1-\alpha\right)^2} \prod_{j\neq i}^{r} \frac{n_i+\beta_i-\beta_j}{n_i-n_j+\beta_i-\beta_j} M_{\vec{n}-\vec{e}_i}^{\vec{\beta},\alpha}(s), \quad (4.3.3)$$

where the right-hand side of (4.3.3) *and the product term* $x(s)M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$ *are defined in* (1.2.14) *and* (3.2.18)*, respectively.*

Proof. From Theorem 3.2.4 we have

$$x(s) M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = M_{q,\vec{n}+\vec{e}_k}^{\vec{\beta},\alpha}(s) + b_{\vec{n},k} M_{q,\vec{n}}^{\vec{\beta},\alpha}(s)$$
$$+ q^{|\vec{n}|}(\alpha q^{|\vec{n}|-1}) \sum_{i=1}^{r} \frac{x(n_i)x(\beta_i+n_i-1)}{\left(1-\alpha q^{|\vec{n}|+\beta_i+n_i-1}\right)\left(1-\alpha q^{|\vec{n}|+\beta_i+n_i-2}\right)}$$
$$\times \prod_{j\neq i}^{r} \frac{x(n_i+\beta_i-\beta_j)}{x(n_i+\beta_i-n_j-\beta_j)} B_{\vec{n},i} M_{q,\vec{n}-\vec{e}_i}^{\vec{\beta},\alpha}(s),$$

where $b_{\vec{n},k}$ and $B_{\vec{n},i}$ are defined in (3.2.18).

Notice that, applying limit in both sides of the above expression when q approaches to 1 and by (3.1.8) we have

$$\lim_{q\to 1} b_{\vec{n},k} = (\beta + |\vec{n}|)\left(\frac{\alpha_k}{1-\alpha_k}\right) + \sum_{i=1}^{r} \frac{n_i}{1-\alpha_i},$$

$$\lim_{q\to 1}\left(\frac{q^{|\vec{n}|}(\alpha q^{|\vec{n}|-1})(q^{n_i}-1)(q^{\beta_i+n_i-1}-1)}{(q-1)^2\left(1-\alpha q^{|\vec{n}|+\beta_i+n_i-1}\right)\left(1-\alpha q^{|\vec{n}|+\beta_i+n_i-2}\right)} \prod_{j\neq i}^{r} \frac{\frac{(q^{n_i+\beta_i-\beta_j}-1)}{(q-1)}}{\frac{(q^{n_i+\beta_i-n_j-\beta_j}-1)}{(q-1)}} B_{\vec{n},i}\right)$$
$$= \frac{\alpha n_i\left(\beta_i+n_i-1\right)}{\left(1-\alpha\right)^2} \prod_{j\neq i}^{r} \frac{n_i+\beta_i-\beta_j}{n_i-n_j+\beta_i-\beta_j},$$

and by Proposition 4.1.3

$$\lim_{q\to 1} x(s) M_{q,\vec{n}}^{\vec{\beta},\alpha}(s) = M_{\vec{n}+\vec{e}_k}^{\vec{\beta},\alpha}(s) + \left[(n_k+\beta_k)\left(\frac{\alpha}{1-\alpha}\right) + \frac{|\vec{n}|}{1-\alpha}\right] M_{\vec{n}}^{\vec{\beta},\alpha}(s)$$
$$+ \alpha \sum_{i=1}^{r} \frac{n_i\left(\beta_i+n_i-1\right)}{\left(1-\alpha\right)^2} \prod_{j\neq i}^{r} \frac{n_i+\beta_i-\beta_j}{n_i-n_j+\beta_i-\beta_j} M_{\vec{n}-\vec{e}_i}^{\vec{\beta},\alpha}(s),$$

which completes the proof of relation (4.3.3). □

Chapter 5
Conclusions and open problems

5.1 Conclusions

We have obtained new families of special functions, namely, the multiple orthogonal polynomials of q-Charlier, q-Meixner (of the first and second kind), and q-Kravchuk. For each of these polynomial family we have obtained the corresponding Rodrigues-type formula (2.3.5), (2.3.9), (2.3.14), and (2.3.18). In addition, for the q-Charlier multiple orthogonal polynomials we found an explicit representation in terms of a new q-analogue of the second of Appell's hypergeometric functions of two variables (3.2.11).

Three of the aforementioned q-multiple polynomial families were found to be common eigenfunctions of two different $(r+1)$-order difference operators, namely, (3.1.9) and (3.2.5) for the q-Charlier multiple orthogonal polynomials. Similarly, difference operators (3.1.18) and (3.2.14), for the q-Meixner multiple orthogonal polynomials of the first kind, as well as difference operators (3.1.31) and (3.2.18), for the q-Meixner multiple orthogonal polynomials of the second kind. For the q-Kravchuk multiple orthogonal polynomials we found just one $(r+1)$-order difference operators (3.1.40). Our limitation on this regard is based on some technical difficulties appeared in the attainment of the recurrence relation.

An important novelty of the algebraic approach developed in this Thesis for the attainment of nearest neighbor recurrence relations (3.2.5), (3.2.14), and (3.2.18) relies on the fact that the requirement of introducing type I multiple orthogonality is omitted, which is a significant difference with respect to the method presented in [92]. We directly proceed from the q-difference operators, instead. In fact, the q-difference operators involved in the Rodrigues-type formula constitute the key-ingredient in our approach making this approach more consistent and algebraically efficient. In this direction, we introduced Lemma 3.3.1 for multiple orthogonal polynomials as a limiting situation of Lemma 3.2.1, when $q \to 1$. Moreover, this fact allows to study the recurrence relations for some known families, namely (3.3.4) and (3.3.10) for the multiple Hahn polynomials as well as for some new situations like the presented unusual case. Indeed, these relations are new in the literature and can be used to define new recurrence relations involving integers, which are relevant in analytic number theory problems [63].

Finally, we show some limiting situations when $q \to 1$, which allow us to recover the corresponding structural relations for multiple orthogonal polynomials of Charlier, Meixner (of the first and second kind), and Kravchuk [10, 50, 65, 66], respectively. More specifically, for the q-Charlier multiple orthogonal polynomials our relations (2.3.5), (3.1.9), and (3.2.5) transform into (1.2.4), (1.2.5), and (1.2.6), respectively. For the q-Meixner multiple orthogonal polynomials of the first kind, the relations (2.3.9), (3.1.18), and (3.2.14) transform into (1.2.8), (1.2.9), and (1.2.10), respectively. For the q-Meixner multiple orthogonal polynomials of the second kind our relations (2.3.14), (3.1.31), and (3.2.18) transform into (1.2.12), (1.2.13), and (1.2.14), respectively. For the q-Kravchuk multiple orthogonal polynomials our relations (2.3.18) and (3.1.40) also transform into (1.2.16) and (1.2.17), respectively.

5.2 Published material

Part of the work carried out for this Thesis has been published in [13] and [14], respectively. Therefore, there is an overlap between this Thesis and the aforementioned publications. In particular, Chapter 2 and Chapter 3 overlap with [13] and [14], in which algebraic properties of q-multiple orthogonal polynomials are studied. In these articles, we study the q-Charlier multiple orthogonal polynomials as well as some algebraic properties for q-Meixner multiple orthogonal polynomials but only for the first kind family. The results involving the q-Meixner multiple orthogonal polynomials of the second kind as well as q-Kravchuk multiple orthogonal polynomials are not yet published.

5.3 Open problems

We propose three problems: the first one concerns to the algebraic theory of multiple orthogonal polynomials while the second one deals with the analytic theory of multiple orthogonal polynomials. The third one is devoted to some applications in quantum physics.

Problem 1. We propose to study some determinants whose entries are orthogonal polynomials [52, 59, 96]. This type of determinants are related to both applied and theoretical researches. See, for instance, in the context of some physical applications [78] for shape-invariant potentials and pseudo virtual states transformations as well as [58] for birth and death processes or the connection with Young diagrams in [40]. Additionally, for theoretical aspects like monotonicity properties of Wronskians see [53], for high order Turán inequalities see [31], and for Selberg-type formulas see [36]. Other recent applications of Wronskians of orthogonal polynomials can be found in connection with random matrix theory in [23, 69].

The Wronskian of orthogonal polynomials is defined by

$$\begin{aligned} W(n,l;x) &= W\left(Q_n(x), Q_{n+1}(x), \ldots, Q_{n+l-1}(x)\right) \\ &= \det \begin{pmatrix} Q_n(x) & Q_{n+1}(x) & \cdots & Q_{n+l-1}(x) \\ Q'_n(x) & Q'_{n+1}(x) & \cdots & Q'_{n+l-1}(x) \\ \vdots & \vdots & \vdots & \vdots \\ Q_n^{(l-1)}(x) & Q_{n+1}^{(l-1)}(x) & \cdots & Q_{n+l-1}^{(l-1)}(x) \end{pmatrix}, \end{aligned}$$

where

$$Q_n(x) = k_n(-x)^n + \cdots, \qquad k_n > 0, \qquad n \in \mathbb{N} = \{0, 1, 2, 3, \ldots\},$$

is a sequence of orthogonal polynomials with respect to an arbitrary measure whose distribution function has an infinite number of increasing points.

In [60], Karlin and Szegö developed a general theory regarding the determinants whose entries are orthogonal polynomials [26, 90]. They showed that:

- If l is even, $(-1)^{l/2}W(n,l;x) > 0$, $x \in \mathbb{R}$. The Wronskian keeps a constant sign for all real x.
- If l is odd, then $W(n,l;x)$ has exactly n simple real zeros and the zeros of $W(n,l;x)$ and $W(n+1,l;x)$ strictly interlace.

Since multiple orthogonal polynomials are natural generalizations of orthogonal polynomials, in [97] the authors considered Wronskian type determinants whose entries are multiple orthogonal polynomials. They have extend for these polynomials the results obtained by Karlin and Szegö for scalar orthogonal polynomials.

In [97] the authors considered a sequence of multi-indices $(\vec{n}_0, \vec{n}_1, \ldots, \vec{n}_{l-1})$, $l \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, such that

- $\vec{n}_0 = \vec{n}$ for a given initial multi-index $\vec{n}$,

- $|\vec{n}_j| = |\vec{n}| + j$ for $j = 1, \ldots, l-1$,
- $\vec{n}_0 \leq \vec{n}_1 \leq \cdots \leq \vec{n}_{l-2} \leq \vec{n}_{l-1}$ (componentwise),

and define the associated Wronskian of multiple orthogonal polynomials by

$$\begin{aligned} W(\vec{n}, l; x) &= W\left(P_{\vec{n}_0}(x), P_{\vec{n}_1}(x), \ldots, P_{\vec{n}_{l-1}}(x)\right) \\ &= \det \begin{pmatrix} P_{\vec{n}_0}(x) & P_{\vec{n}_1}(x) & \cdots & P_{\vec{n}_{l-1}}(x) \\ P'_{\vec{n}_0}(x) & P'_{\vec{n}_1}(x) & \cdots & P'_{\vec{n}_{l-1}}(x) \\ \vdots & \vdots & \vdots & \vdots \\ P^{(l-1)}_{\vec{n}_0}(x) & P^{(l-1)}_{\vec{n}_1}(x) & \cdots & P^{(l-1)}_{\vec{n}_{l-1}}(x) \end{pmatrix}, \end{aligned} \tag{5.3.1}$$

where $P_{\vec{n}}$ is the type II multiple orthogonal polynomial.

Some of their main results obtained in this context are

Theorem 5.3.1. *Suppose that the weights $(w_1, w_2, \ldots, w_r)$ form an AT system on $[a, b]$ for all the multi-indices $\vec{n} \in \mathbb{N}^r$, then we have*

$$W(\vec{n}, l; x) > 0, \qquad x \in \mathbb{R}, \tag{5.3.2}$$

if l is even, where $W(\vec{n}, l; x)$ is defined in (5.3.1).

Theorem 5.3.2. *Let $w_1, w_2, \ldots, w_r$ be r weights as in Theorem* 5.3.1 *and let l be odd. For each fixed multi-index $\vec{n}$ the polynomials $W(\vec{n}, l; x)$ have exactly $|\vec{n}|$ simple zeros on the real axis. Furthermore, given two paths consisting of l multi-indices such that the last $l-1$ multi-indices of one path starting from $\vec{n}$ coincide with the first $l-1$ multi-indices of the other path ending at $\vec{m}$ (which also means $|\vec{m}| = |\vec{n}| + l$ and $\vec{n} \leq \vec{m}$), then the real zeros of two associated Wronskians strictly interlace.*

Based on the above results the following question arises: Theorems 5.3.1 and 5.3.2 can be extended for the q-multiple orthogonal polynomials studied in this Thesis? We expect a positive answer because most of the techniques used by the above authors can be extended in a natural way in our context. In this direction, some studies on Casorati determinants have been already conducted in [35] about some symmetries for these determinants involving classical discrete orthogonal polynomials and in [79] about Casoratian identities for Wilson and Askey-Wilson polynomials. Recently, in [37] the authors started the study of Casorati type determinants of some q-classical orthogonal polynomials studied in [3, 61].

Problem 2. A description of the main term of the logarithm asymptotics of q-multiple orthogonal polynomials deserves our future attention. We expect to give it in terms of an algebraic function approach (see [6]). These results yield the Cauchy transform of the weak-star limit for scaling zero counting measure of the polynomials. In addition, the zero distribution of these type II q-multiple orthogonal polynomials should be studied and the generalization of the recurrence coeficcients. More precisely, we propose to study the weak asymptotic behavior of q-discrete multiple orthogonal polynomials, whose study is still missing in the literature (see [15] for asymptotic analysis of classical discrete orthogonal polynomials). For such a purpose we will use an algebraic function formulation for the solution of the equilibrium problem with constraints [18, 34, 48] to describe the zero distribution of such multiple orthogonal polynomials [84]. This approach has been recently developed for multiple Meixner polynomials in [6] (see [8] as well as [67] and [86] for orther approaches). Moreover, by analyzing the limiting behavior of the coefficients of the recurrence relations for such polynomials we expect to obtain the main term of their asymptotics.

Other direction of research could be the ratio asymptotics of multiple orthogonal polynomials. In [72] the asymptotics for the ratio of multiple Charlier polynomials is obtained. This result will help us to address a similar question involving the new families of q-multiple Charlier polynomials studied in Chapter 4. In addition, the Riemann-Hilbert approach to global asymptotics of discrete orthogonal polynomials with infinite nodes was studied in [80], which opens a new direction for future studies involving q-multiple orthogonal polynomials.

Problem 3. Some applications of multiple orthogonal polynomials have been studied in quantum physics. In [70, 71] the authors develop a physical model involving multiple polynomials of Charlier and Meixner of the first type. For the Meixner case the physical model was summarized in the section 1.4.3. However, there is a similar model for the multiple Charlier case that we will explain below. Indeed, the authors in [71] consider the set of r Hamiltonians $H_i^{\vec{\alpha}}, i = 1, \ldots, r$, defined as follows

$$H_i^{\vec{\alpha}} = \sum_{j=1}^{r} a_j^{+} a_j + \sum_{j=1}^{r} \sigma_j a_j^{+} + a_i + \sigma_i, \quad i = 1, \ldots, r,$$

where the operators $a_j^{+} a_j$, a_j^{+} and a_j are defined in the section 1.4.3.

The multiple Charlier polynomials simultaneously diagonalize the r non-Hermitian oscillator Hamiltonians. Consider the states $|x, \vec{\alpha}\rangle$ defined by means of the combination of states $|n_1, \ldots, n_r\rangle$ as:

$$|x, \vec{\alpha}\rangle = N_{x,\vec{\alpha}}^{r} \sum_{\vec{n}} \frac{C_{\vec{n}}^{\alpha}(x)}{\sqrt{n_1! \cdots n_r!}} |n_1, n_2, \ldots, n_r\rangle, \quad x \in \mathbb{N}.$$

Thus,

$$\begin{aligned} H_i^{\vec{\alpha}} |x, \vec{\alpha}\rangle = N_{x,\vec{\alpha}}^{r} \sum_{\vec{n}} \frac{1}{\sqrt{n_1! \cdots n_r!}} \Big[& C_{\vec{n}+\vec{e}_i}^{\vec{\alpha}}(x) + (\sigma_i + |\vec{n}|)\, C_{\vec{n}}^{\vec{\alpha}}(x) \\ & + \sum_{j=1}^{r} n_j \sigma_j C_{\vec{n}-\vec{e}_j}^{\vec{\alpha}}(x) \Big] |n_1, n_2, \ldots, n_r\rangle. \end{aligned}$$

Now, by using the recurrence relation for multiple Charlier polynomials (1.2.6) one obtains

$$H_i^{\vec{\alpha}} |x, \vec{\alpha}\rangle = x |x, \vec{\alpha}\rangle.$$

Although operators are non-Hermitian, they have a real spectrum given by the non-negative integers. The states $|x, \vec{\alpha}\rangle$ are uniquely defined as the joint eigenstates of the Hamiltonian operators with eigenvalues equal to x. Moreover,

$$[H_i^{\vec{\alpha}}, H_j^{\vec{\alpha}}] |x, \vec{\alpha}\rangle = 0.$$

However, these Hamiltonians do not commute pairwise. Indeed,

$$[H_i^{\vec{\alpha}}, H_j^{\vec{\alpha}}] = a_i - a_j + (\sigma_i - \sigma_j)\,.$$

In closing, because they do not commute and yet have common eigenvectors, we say that they form a 'weakly' integrable system.

Finally, we propose to extend the techniques used by the above authors for multiple orthogonal polynomials of q-Charlier and q-Meixner of the first kind, respectively, studied in this thesis.

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